A Game-theoretic and Algorithmic Study of the Toll Rates of Hong Kong Road Tunnels

by

ZHANG Zhili (UID: 3035141243)

supervised by

Dr. HUANG Zhiyi

in the

Department of Computer Science

of the

University of Hong Kong

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Abstract

Traffic networks in Hong Kong have been suffering from extensive congestion for years, especially areas near the three harbor tunnels connecting Kowloon and Hong Kong Island. A specific pattern of congestion emerges that one of the three tunnels, the Western Harbor Crossing, is usually free of congestion, due to a probable factor of its relatively high toll rates compared with the other two.

In this project, the specific congestion issue occurred at three harbor tunnels in Hong Kong and their peripheral areas will be focused. Modeling will be carried out in transforming the real congestion problem into congestion game discussed in Game Theory areas. Based on this, the algorithmic problem of computing the best toll rates, potentially in both the offline and online settings, will be further explored.

On the lower level, works will be carried out mainly on designing a network model that reasonably describes the actual features of traffic conditions around three harbor tunnels in Hong Kong, and effectively reflects their performance provided with drivers' behaviors. On the upper level, efforts will be invested on the design of algorithm that computes the toll rates in reducing traffic congestion theoretically. A demonstration is also expected in proving the utility of our modeling and algorithm as well.

At the time of writing, modeling of the congestion game is ongoing. Particularly the features of the tunnels' traffic network are being investigated.

Acknowledgement

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Abbreviations

CHT	The Cross-Harbor Tunnel
EHC	The Eastern Harbor Crossing
WHC	The Western Harbor Crossing
PoA	Price of Anarchy

Chapter 1 Introduction

1.1 Problem specification

With the assistance of GPS system and audio broadcast offering real-time traffic information, congestion still seems an insurmountable urban issue prevailing in most metropolises around the world. Specifically, a pattern of congestion is noticeable that those bridges or tunnels that locates at crucial position in a city, connecting two major areas for example, are more likely under a constant congestion throughout daytime.[1]

Here in Hong Kong, three major tunnels in parallel connects Hong Kong Island to Kowloon. Specifically, the Cross-Harbor Tunnel and the Eastern Harbor Crossing are more likely suffered from congestion than the Western Harbor Crossing. From the perspective of theoretic computer science, this system has a relatively low efficiency as the capacity of WHC is not effectively used while CHT and EHC are exploited. This phenomenon could be ascribed to the relatively higher toll rates issued by WHC, twice as the rate issued by EHC and three times as the one issued by CHT. [7]

The difference in toll rates contributes to the uneven flow distribution among harbor tunnels. This motivates our project to model the real congestion issue as the congestion game in the field of Algorithmic Game Theory.[2][5] Specifically, we might draft a start-destination flow network based on geographical reality of harbor tunnels in Hong Kong, see the drivers as rational players and treat the toll rates as a controllable factor among all the factors that incurs a cost onto the players. By manipulating the toll rates, the modeling could potentially lead us to a theoretical solution of congestion, an impartial distribution of driving flows consequently.

1.2 Previous works

Previous researchers have made remarkable effort in exploring the relevant problems. Richard Cole and his colleagues developed a theory on the modeling of pricing network problems, which suggested a solution in the general form and proved its feasibility provided with certain conditions.[4] This result provides a theoretical base and motivates the feasibility of our project to a certain extent. Besides, Hai YAN and his colleague from civil engineering field discussed the possibility of applying road tolls as a general strategy in tackling queueing and congestion problems, and they also proposed a reasonable algorithm in computing the road tolls and proved the stability of their algorithm under some simple cases.[1][3] While these works have suggested from theoretical perspective the feasibility of controlling congestion traffic with toll rates, it should be noticed that few work was found in applying the relevant theories to a specific congestion problem with the modeling approach, probably as the consequence of the complexity incurred by modeling a real routing problem into a flow network with agents.[6]

1.3 Scope

This project focuses on the specific case of the congestion problem and uneven driving flow occurred among harbor tunnels connecting Kowloon and Hong Kong Island. Especially the major effort will be devoted in the design and enhancement of the modeling of the flow network. The reason is two-fold. On the one hand, the three parallel tunnels together with their peripheral areas are of a geographically simple pattern compared with any specific road network of a city. As the tunnels stand out to be a relatively independent existence, it makes it easier to enhance the quality of our model without considering the triviality of traffic. On the other hand, the solution is expected to mitigate the congestion problem in harbor tunnels. By concentrating on the application of algorithmic game theory on reality, the model is anticipated to be refined as closed to the actual performance as possible. A feasible toll-rate computing method is also a desirable outcome of the project.

1.4 Outline

This report will unfold as follow. In chapter 2, the previous works regarding the problem will be presented and discussed in the form of literature review. The methodology of the project will be delivered later in Chapter 3, with detailed demonstrations of the approaches taken. Chapter 4 mainly discusses the results of the project and illustrates the feasibility of the proposed algorithm in tackling the problem. It also offers details about a model that was evaluated during the progress. And this is followed by the limitation of our project and the difficulties we have encountered in chapter 5. The brief conclusion will be given in chapter 6 at last.

Chapter 2 Literature Review

2.1 Congestion Pricing Policies in History

Traffic congestion has been a pervasive problem in many urban areas around the globe. Aiming to tackle this issue, various policies have been introduced and implemented by different local governments. An empirical ideal solution suggested and supported by many policy analysts and economists would some form of congestion pricing.[10] Economics researcher Kenneth A. Small from the University of California, Irvine, and researcher Jose A. Gomez-Ibanez from Harvard University have given a detailed review and analysis of congestion-controlling policies applied around the world in their published paper in 1998.[10] The paper offers a fresh scope in understanding those congestion-pricing policies applied at that time, specifically under what scenario a policy is generated and applied, how it works theoretically and practically, and how effective it was in controlling congestion.

Three cities, Singapore, Hong Kong and Cambridge, England have been the pioneers in applying congestion pricing of city centers, each implementing the concept with a unique form of pricing scheme. Singapore was the first among them, implementing a relatively simple yet practical scheme called “Area License Scheme” (ALS). “The priced area for ALS was defined by a single cordon line surrounding the city center”, as is introduced by Kenneth in his paper. Fee was imposed initially on vehicles entering this restricted area during morning peak hours only. In 1989 and

1994, for the purpose to deal with a more severe situation of congestion during other daytimes, the priced time period was extended twice to include the afternoon peak and the time between morning and afternoon peak respectively. The implementation of ALS on traffic congestion was effective[10]. As is mentioned by Kenneth, the number of vehicles entering the restricted zone during priced period declined by 44 percent. However, as the fee set by ALS was relatively high, researchers have been posting doubt onto ALS. They could hardly accept a "crude" scheme as ALS which impose almost only a strong high-pricing incentives to commuters to be a "good idea".[10] Hong Kong, at the time in 1985 with 4 million population, was also suggested with three different road pricing schemes in controlling congestion. Similar with the ALS in Singapore, the three schemes set up several priced zones; they differed mainly in their pricing target of commuting vehicles. The three pricing plans were predicted to be effective, yet none of them were adopted for a combined reason. Cambridge, a historic city in England, was proposed a congestion-specific charging scheme. Charges within the city center would vary in real time according to the amount of congestion experienced by the individual vehicle, as is suggested by Sharpe in 1993.[10] The scheme was a fresh and innovative policy as it was proposed in 1990s, and it is also close to the theoretical extreme. Unfortunately, the scheme did not get enough support from either the public or the politicians, because the public were worried about the unpredicted amount of fees they would be charged, and the politicians preferred a relatively conservative policy with empirical testing.

Kenneth in his paper in 1998 also covered other forms of pricing schemes. California introduced a unique pricing scheme for a congested commuter route, the State Route 91. The scheme added two priced lanes in each direction called "91 Express Lanes". This project received generally favorable feedbacks in surveys, as is mentioned[10], with more than half of the public approving its toll finance and the time-varying tolls. The Randstad region of the Netherlands was proposed a huge road pricing system to deal with congestion, involving multiple cordon system with more than 100 charging points and even time-varying tolls. The proposal focused on technology and modeling of predicting traffic impacts. However, it was such a complicated and gigantic scheme that it was easily questioned with technical feasibility and other issues combined with the implementation of the scheme. The scheme fails to persuade the doubtful public, and was substituted with a relatively modest and conventional plan of road pricing.

Throughout Kenneth's paper, various congestion-pricing schemes, each with their unique settings and backgrounds, were thoroughly discussed. It was shown by the study as well as the actual experience that congestion pricing is substantially effective in reducing traffic congestion. However, the enforcement of a specific scheme is far more than a detailed theoretical analysis and prediction of effectiveness. The degree of public understanding towards the scheme and the supports and innovations of politicians are more important factors in pushing the scheme forward to implementation.

2.2 Past Theoretical Researches

Confirming with the belief of congestion-pricing's effectiveness, more efforts were spent on the discovery of previous research works in analyzing the theoretical problem.

In the paper written by Yaron Hollander and Joseph Prashker, published in 2006, the authors provided a steady proof of the applicability in applying non-cooperative game theory (NCGT) in transport analysis[5]. They classify games into groups in two ways. One way is to categorize games into concept games and instrument games. While the concept games focus on a specific part of small-scale in a great problem, serving to “establish theoretical principles”, the former, instrument games, are applied to real scenarios and are relatively application-oriented. The other categorization classifies games into following groups: games against a demon; games between travelers; games between authorities; and games between travelers and authorities. This kind of grouping serves as a guidance in analyzing and choosing an appropriate model based on the specific scenario. Yaron then presented review and analysis, based on the second categorization, of the four groups of NCGT models, with the instances of models proposed by previous researchers. Yaron provides a generalization and his own insights for each group of models, and support the opinion that NCGT models have potential for further theoretical analysis on traffic problems.

Chapter 3 Methodology

In this chapter the approaches of the project will be introduced and rationalized. The implementations for a concrete model will be discussed, followed by the expected cascading development of our model.

The project is mainly based on theoretical modeling of actual cases in Hong Kong harbor tunnels. In the introduction part it has already been shown that the variances of toll rates among different paths result in the uneven distribution of driving flows. This indicates a potential availability of controlling driving flows' distribution with meticulously calculated toll rates, and it motivates the approach, having toll rates as the major variable in the model. Based on the choice of major variable, and the illumination of work conducted by previous researches, the congestion game is an ideal candidate of the model. Before the specification of modeling, a few concepts and theorem will be introduced in advance.

3.1 Introduction of concepts and theorems

3.1.1 Definitions and Concepts

Potential game [9]- In game theory, a game is said to be a potential game if the incentive of all players to change their strategy can be expressed using a single global function called the potential function.

Congestion game [9]- Congestion game is a kind of potential game, in which we define players and resources. The payoff of each player depends on the resources which the player chooses, and the number of players choosing this same resource (in other words, how congested the resource is).

Nash equilibrium [11]– Nash equilibrium in game theory is a solution concept of a non-cooperative game involving two or more players. In Nash equilibrium, each player is assumed to know the equilibrium strategies of the other players, and no player benefits from changing his own strategy.

Pure strategy and pure Nash equilibrium [11]- Pure strategies are the strategies when every player chooses the same strategy each time. It contrasts with mixed strategies when player has a set of probabilities defined for his strategies. Pure Nash equilibrium is Nash equilibrium containing only pure strategies for every player. The opposite is mixed Nash equilibrium.

Selfish routing [8]– Selfish routing is a concept first introduced by Tim Roughgarden in his thesis in 2002. It defines, in a congestion game, a player's selfish, uncoordinated behavior.

Non-atomic selfish routing – non-atomic selfish routing is a model of routing that assumes there are very large number of players, each controlling a negligible fraction of the overall traffic. This contrasts with atomic selfish routing, in which each player controls a nonnegligible amount of traffic.

Price of Anarchy [8]– Price of anarchy (PoA) is a concept in game theory that measures how efficiency of a system degrades due to selfish behavior of its agents. As far as the project concerns, the price of anarchy of a congestion game could be defined as follow:

$$\text{POA} = \frac{total\ cost\ of\ routing\ in\ equilibrium\ case}{total\ cost\ of\ routing\ in\ optimal\ case}$$

Pigou-like Network - A Pigou-like network (see Figure 1) has:

- Two vertices, s and t.
- Two edges from s to t.
- A traffic rate $r > 0$.
- A cost function $c(\cdot)$ on the first edge.
- The cost function everywhere equal to $c(r)$ on the second edge.

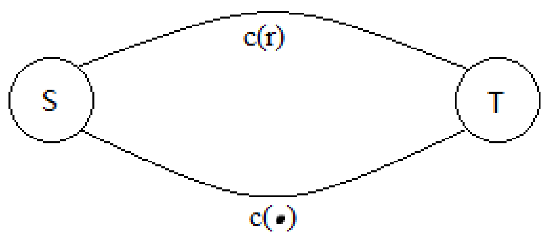

Figure 1. Pigou-like network

3.1.2 Theorems

Theorem 1 Rosenthal's Theorem [4]

Every atomic selfish routing game, with arbitrary real-valued cost functions, has at least one equilibrium flow.

Theorem 2 Tight POA Bounds for Selfish Routing [4]

For every set $\mathcal{C}$ *of cost functions and every selfish routing network with cost functions in* $\mathcal{C}$*, the POA is at most* $\alpha(\mathcal{C})$ *where:*

$$\alpha(\mathcal{C}) := sup_{c\in\mathcal{C}}\, sup_{r\geq 0}\, sup_{x\geq 0} \left\{\frac{r\cdot c(r)}{x\cdot c(x)+(r-x)\cdot c(r)}\right\}$$

3.2 Construction of model

This project models the congestion problem of Hong Kong harbor tunnels as a congestion game. Specifically, the three tunnels connecting Hong Kong Island and Kowloon, which are CHT, EHC, WHC respectively, are modeled as resources. Drivers passing the tunnels are modeled as the player who apply selfish routing as their behavior. As for the payoff function, in other way the cost function, of a resource, the toll rates and other relevant factors will be taken into consideration.

This kind of modeling is reasonable as it was applied in previous researches.[1] More importantly, it can relatively reflect the actual case of the congestion problem in the harbor tunnels while generalizing the real-life problem to the degree to be theoretically analyzed. A primary reason is that the three harbor tunnels being a relatively independent existence from their peripheral traffic are suitable to be considered as three individual paths in a network. And this fact extremely facilitates and simplifies the further calculation and computation, as it eliminates a common difficulty faced by a modeling of real-life traffic network, the complexity of geometrical frame of the network.

The reason of applying selfish routing as the model for drivers, the players in this game, is also straightforward. A normal driver would prefer a route with fewer toll charge to one with higher charges, and prefer a relatively fast, lower-time-consumed route to an opposite. This generalizes the selfish behavior of a real driver, which corresponds to the model of selfish routing. Moreover, a normal driver has little information about other drivers, and it is difficult, and even impossible for them to decide what influence their individual driving behavior would exert onto the entire traffic network, due to their little awareness of the entire, macro traffic. And this leads to a conclusion that no substantial coordinating behavior would exists among drivers in a traffic network. In this way, the selfish routing is ideal and most appropriate to describe the driving behavior of all the players in this game.

As for the payoff function, the toll rates applied to the tunnels, the congestion on the tunnel, as well as other factors concerning the traffic, are generalized as cost functions incident to paths in the network of the model. Driving is a kind of subjective experience for an individual, and it is seldom evaluated as an index to the driver. However, we can still try to quantify payoff of the driving experience of a driver. The payoff could be conceptually generalized as the below formula:

$$Payoff = Achievement - Cost$$

"Achievement" is simply the completion of the journey of a tunnel. In this project, we model this index as a constant 1, representing the maximum payoff that a driver can get by having this journey.

"Cost" is the index representing all the factors that negatively contributing to the driving experience. It is concerned with the congestion that a driver experiences, the toll that a driver was charged, as well as other possible factors. For the purpose of simple computation, the formula of linear combination between congestion and toll rates is applied as below:

$$Cost = Congestion + (constant) * Toll$$

As the ultimate target of the project is proposing a reasonable toll rate-computing algorithm in mitigating congestion, it is crucial to set up a concrete and basic scenario of the congestion game, based on which we can elaborate the game theory computation and further design and test our algorithm. Here we propose a congestion game scenario, in which there are 100 players commuting on 2 parallel independent tunnels. We also assume that the entire scenario lasts for 50 days. For each day, the driver will choose a relatively ideal path based on his knowledge of the two tunnels, including the congestion he experienced "yesterday" as well as the toll rate he will charged "today". In the meantime, we could possibly control and modify the daily toll rates and explore how the 100 players would shift or stick their choice based on their knowledge.

3.3 Further Discussion on Payoff Function

As is indicated in the project title and the introduction, toll rate is the primary factor used in this project in controlling and mitigating congestion. And it is one of the major concern of this project that, before the toll rates are decided and applied, it should be primarily investigated how the toll change would influence the equilibrium of the game, the outcome of players' behavior. Mentioned in section 3.2, toll rate is modeled as a contributing factor to the payoff function. It would occur as a natural thought to compare the equilibria of the games with and without toll rate as a contributing factor.

In accordance with this logic, the processing of model would unfold in the following way. First the model without toll rate will be investigated and the equilibrium under this circumstance will be computed. Then we evaluate the case involving an unchanged toll rate, generalize the toll as a constant term in the payoff function. The corresponding performances of players under the updated circumstance will be re-computed and compared with previous result. Then, instead of a constant value, we will try having toll rate as a variable which adapts to the congestion of two paths. With this setting, we will explore how the players choices will trend, as well as the theoretical equilibrium.

The payoff formula for a player has been given in the previous chapter as:

$$Payoff = Achievement\ (= 1) - Congestion - (constant) * Toll$$

In this formula, we set up the upper bound for the payoff to be 1. And we use the linear combination of the "congestion" and the "toll" to represent the cost that a driver suffers. However, there is substantially no way for us to deduct a precise formula of the two factors relative to the constant Achievement. The reason is straightforward. Because it is hardly possible to compare the quantitative value between the concepts of Achievement and the Congestion, or to compare the concept Congestion with a concrete value “toll rate”. Yet, previous works from mathematicians have suggested ways representing congestion in different mathematical formula. One simple representation is that, the congestion is directly proportional to the players sharing the same resource.

$$\text{Congestion } \propto \text{Sensitivity} * \frac{\text{Number of player}}{Total\ Number\ of\ players}$$

Another representation is that, the congestion is inversely proportional to the difference between maximum utility of the resource and the number of players sharing.

$$\text{Congestion } \propto \frac{Sensitivity}{Maximum\ Utility - Number\ of\ Players}$$

Both methods of representation are reasonable and simple for computation. As a result, they will be applied in parallel and for our easy computation.

What to be noticed is that the concept “Sensitivity” is a value we applied in our computation of congestion. In most scenarios, value of “Sensitivity” is set to be 1. However, it comes to our notice that, although the toll rate with a monetary representation of value, is naturally equivalent to every player, the concept of “Congestion” is valued diversely among different drivers in real life. Some people are more sensitive to congestion, which means that they may suffer a greater loss (i.e. a relatively lower payoff) if they encounter congestion; others could be inert to the congestion. At a later stage of our project, we expect to have different values of sensitivities among all players. And in that way, we hope to evaluate how the equilibrium would shift based on specific scenarios.

As for the toll rate, we expect it to be a reasonable-scaled value compared with the Achievement (=1). Hence, we introduce the constant multiplying with the toll rate to be initially 0.01. And the toll rate, in accordance with the constant, would generally be a value in the range of (1, 100).

With the proposed scenario and the proposed formula, we could summarize the modeling of payoff function with a simple example. Assuming the congestion is directly proportional to the number of players, the numbers of players choosing two tunnels respectively are 40 and 60, and the toll rates charged by two tunnels are 25 and 15. Then, the payoff values for a driver passing through two tunnels are:

$$\text{Payoff}_1 = 1 - \frac{40}{100} - 0.01 * 25 = 0.35$$

$$\text{Payoff}_1 = 1 - \frac{60}{100} - 0.01 * 15 = 0.25$$

It would be noticed that, throughout the process of deducing payoff formula, we have based our formula on many hypothetical setting and assumptions. Admittedly it could hardly be proved that both the congestion and the toll rate contribute linearly to the payoff function. However, as a bold exploration of solutions to the congestion problem, the hypothetical scenario proposed by the project is a reasonable and experimental attempt in testing our design of toll rate algorithm. Also, we could try adjusting the values of data in our original scenario, and cross validate the feasibility of our modeling of the problem.

3.4 Summary of methodology

To conclude all the methods mentioned in this chapter, hereby we present the mathematical modeling of the congestion problem, and we state in what ways we develop our model.

Basic Scenario:

100 Players making selection from two different tunnels in 50 days. For each day, a player chooses the relatively ideal path based on his congestion situation yesterday, as well as the toll rates posted to them in advance.

The payoff function applied by a player is defined as:

$$\text{payoff} = 1 - \text{ sensitivity} * \text{congestion} - \text{C} * \text{ toll rate}$$

Based on the above formula, we propose variations regarding each factors in the formula, so as to increase the complexity of our model. In accordance with the model, we compute the players choices on each day throughout the entire 50-day period. We will also examine how the players' choices as well as the equilibrium shifts in the time period. Based on different settings of toll rate we propose, we will check if the toll rates is helpful in controlling congestion and balancing players' choice distribution among the two.

Variations applied in modeling regarding each factor

1 Toll rates

   1.1 Model without toll rates

   1.2 Model with fixed rates

   1.3 Model with rates that increment/decrement in accordance with players' choices

   1.4 Model with toll rates that are upper/lower bounded

2 Congestion formula

   2.1 Proportional to number of players

   2.2 Inverse proportional to the remain utility of resource.

Chapter 4 Project Result

This chapter is to reveal the experimental result of the project

In accordance with the modeling scenarios – variations of payoff function proposed in section 3.4, the code regarding each scenario was programmed in python.

Specifically, as for the variation 1.1 “Model without toll rates”, the mathematical results of the according equilibrium and players’ choices are relatively straightforward, and they will be calculated and presented directly. As for the each of the rest variations, we design three parallel program iterations, with respectively the same data settings but varied data values. For each iteration, the players’ choice-distribution was computed on each day and recorded. Based on these data, a line chart is draft in order to show the trend of players’ choices as well as the potential equilibrium. Discussion will be given at the end of each variation’s experimental result.

```python
"""
pay-off function
"""
def payoff_func(ratio, sens, toll, constant):
    return (1 - ratio * sens - constant * toll )

'''
Class used to handle the players. Each player has its
'''
class player(object):...


'''
Path class that is used to handle a path's attribute
Has two factors: num_player, toll
'''
class path(object):...


def num_of_players(players, choice):
    return len(filter(lambda x : x.path() == choice, players))


if __name__ == "__main__":...

################################################################################

    initial_dist = [(100, 0), (50, 50), (0, 100)]

    for j in range(3):

        players = [player(i, 0.1, 1, 0) for i in range(initial_dist[j][0])] + [player(i, 0.1, 1, 1)

        paths = [path(initial_dist[j][0], 20), path(initial_dist[j][1], 40)]

        output = open("output/output_initial_dist_%d.txt" % j,'w')
        output.write("Days, Payoff 0, Payoff 1, Players_0, Players_1, Ratio_0\n"     )

        for day in range(50):

            p_0 = payoff_func(ratio(paths[0].num()), 1, paths[0].toll(), 0.01)
            p_1 = payoff_func(ratio(paths[1].num()), 1, paths[1].toll(), 0.01)

            output.write("%d, %.2f, %.2f, %d, %d, %.2f \n" %
                (day, p_0, p_1, paths[0].num(), paths[1].num(), ratio(paths[0].num())) )

            for i in range(100):
                if players[i].path() == 0:
                    if p_0 < p_1:
                        players[i].switch_path()
```

Figure 2. A Glimpse of code

4.1 Model of variations in Toll Rates

4.1.1 Model without Toll Rates

If the toll rates is universally 0, then players' payoff is solely dependent on the congestion. Accordingly, the players' choices only depends on the yesterday's congestion they experienced. We will see that the flow of drivers shifts between two paths. For example, if, at Day 0, the distribution of drivers among two paths are (90, 10), then at Day 1, all those 90 players choosing path-1 will shift their choice, while the other 10 originally choosing path-2 will stick, resulting in the distribution of (0, 100). At Day 2, the driving flow will shift to path-1, resulting in the dist – (100, 0). We can envision that this setting is meaningless as players always crowds to the other vacant path. And this result stimulates us to add an important factor to our model – the probability of a player switching his choice given that the other choice is more ideal, which we initially set to be 0.1. Then, redoing all of the above process, we get the line chart as below:

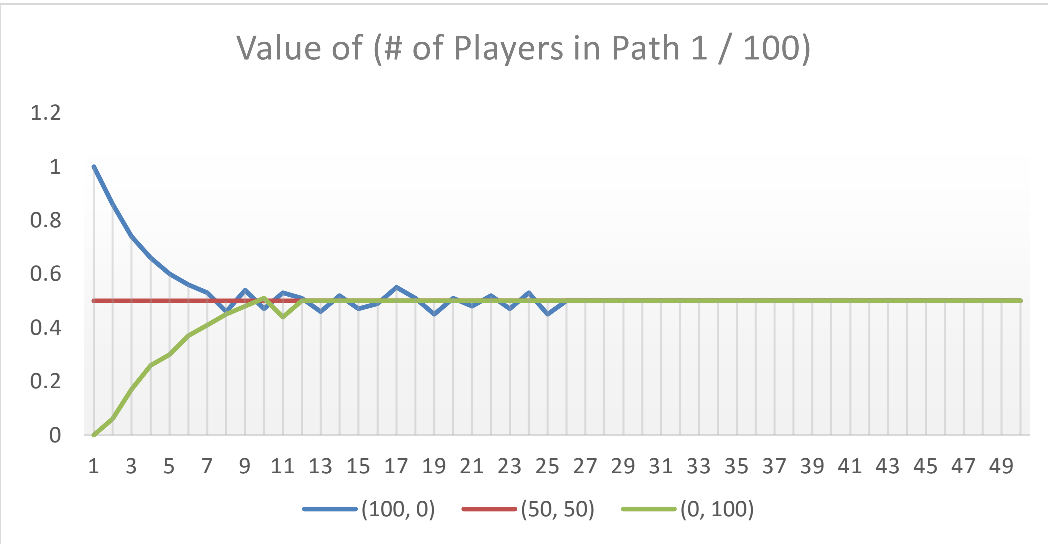

Figure 3 Line Chart - Without Toll rate

Under three scenarios in which the initial distributions are (100, 0), (50, 50), (0, 100) respectively. Unsurprisingly, the ratio of number of players choosing Path-1 converges to 0.5

4.1.2 Model with Fixed Toll Rates

For this variation of payoff function, we apply a fixed toll rate for two paths. To be noticed, we on purpose set distinct toll rates for the two. The reason is two-fold. In real case, the two tunnels EHC and WHC has different pricing scheme, which contributes greatly to the different scale of driving flows on the two tunnels. It would be reasonable if we set the tolls to be different in our model. Also, if two fixed tolls are equal, then it is mathematically equivalent to the previous chapter's model.

At the moment, we need to clarify in advance that the initial distribution of players' choices would have no effect on the converged result, which is also the relative "equilibrium". Nor would the choice-switching probability of a player. To verify these two statements, we designed two iterations of tests.

In the first iteration, we control the initial distribution of players to be (100, 0), (50, 50), (0, 100) respectively. With Path-1's toll rate of 20 and Path-2's rate of 40, and the switching probability of a player to be 0.1, we get the following line chart.

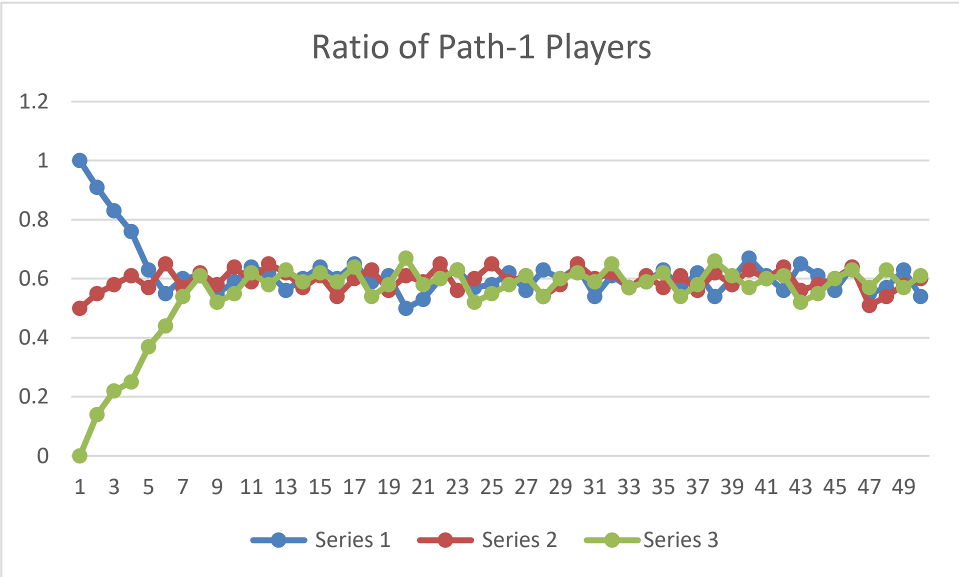

Figure 4 Line Chart - Control Initial Distribution

We see that three lines converges to the similar value around 0.6. If we select the data from Day 10 to Day 49 and calculate their average values, we get 0.596, 0.596 and 0.594 respectively. Indicating that initial distribution has no influence on converged result.

In the next iteration, we control the switch probability to be 0.05, 0.10 and 0.15 in three rounds. The line chart was obtained as below:

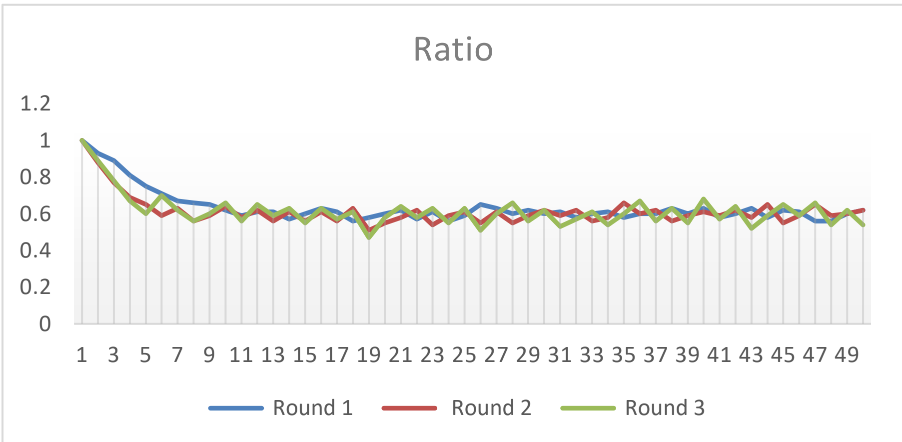

Figure 5. Line Chart - Control Probability

The average of the three set of data are 0.60, 0.59 and 0.59, which proves our hypothesis that switching probability has little effect on equilibrium. We further calculated the standard deviations of the three set of data from Day 10. It turned out that the values of STDEV are 0.022, 0.033, and 0.049. This points out that a larger probability indeed increases the chance of a player switching his choice when there is a relatively more ideal one. And this results in a relatively "unstable" equilibrium after the players' choice distribution converges.

After verifying the two statement beforehand, we could run the iteration controlling toll rates. Specifically, we fix Path-1's rate to be 20, and control Path-2's rate to be 30, 40 and 50 in three rounds respectively. The resulting line chart as follow:

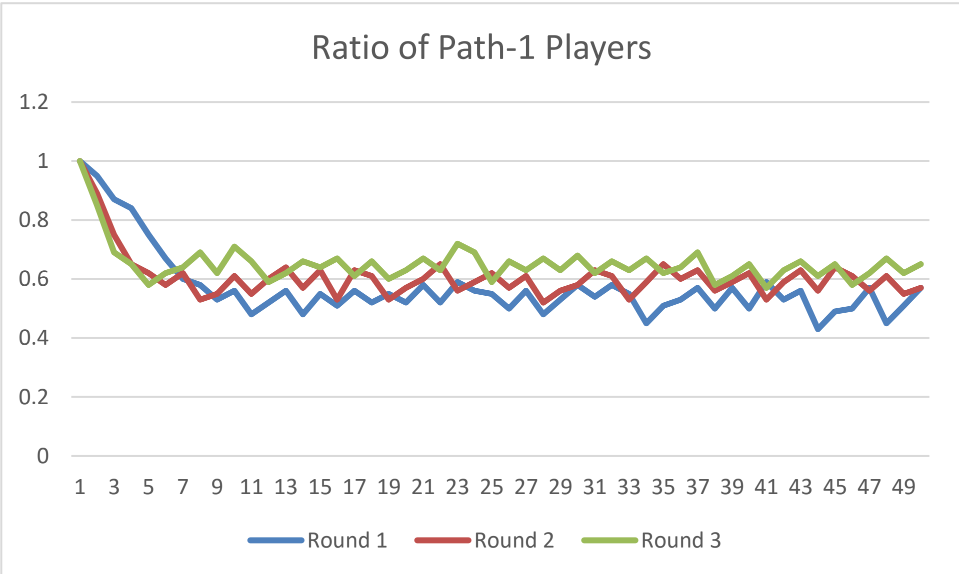

Figure 6. Line Chart - Fixed Toll rates

We could visually observe a different converging results among three rounds. The green line representing round 3 is roughly above red line, which lies above the blue one. The average values of three set of data after Day 10 are computed in the meantime as: 0.53, 0.59 and 0.64.

The result is reasonable as the increase of Path-2's toll rates would promote the advantage of Path-1 being a relatively cheap way. As the Path-2's toll increases, the equilibrium shifts gradually to a relatively uneven state.

4.1.3 Model with Congestion-driven Toll Rates

Based on the result we acquire in previous result, we further modify our model by introducing a new mechanism – the increment and decrement of toll rates. Specifically, if the driving flow on a path is less than the other, the toll rate will decrease for a fixed ratio; the other will increase its rate for a fixed ratio. In three rounds of testing, the ratios are set to be 0.02, 0.05 and 0.10.

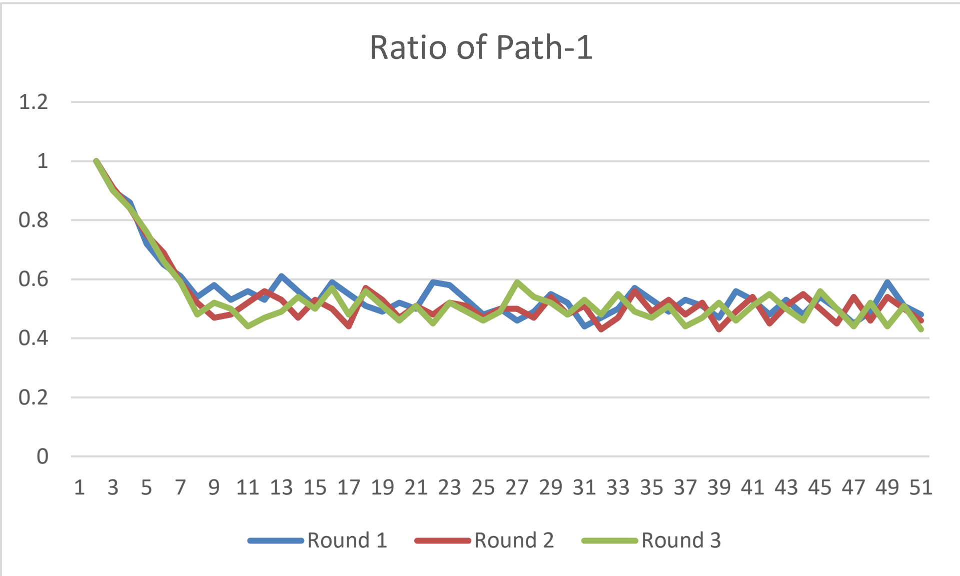

Figure 7. Line Chart - varied toll rates

The ratios in three rounds visually converges to some value around 0.5. The average ratios of three rounds from Day 10 are 0.51, 0.49 and 0.49. A tiny distinction emerges, which triggers the curiosity. We further extended the experiment from 50 days to 100 days, and computed the average ratio of the latter 50 days. The values are 0.501, 0.496 and 0.503. This indicates that the equilibrium will still converges to the same despite the difference of increment/decrement ratio. The further calculation of standard deviations of the latter 50 days' data also points out that a larger daily increment/decrement ratio may result in a relatively unstable equilibrium, even though it fixes and mitigates the congestion problem faster than a smaller daily increment.

We then discuss the effectiveness and feasibilities of applying the congestion-driven toll rates with fixed increment ratio. The ratio of Path-1's players number converges to 0.50 in equilibrium. We then draft a line chart of Path 1 and Path 2's toll rates in round 1. Unsurprisingly, both rates converge to a value around 27. However, as we

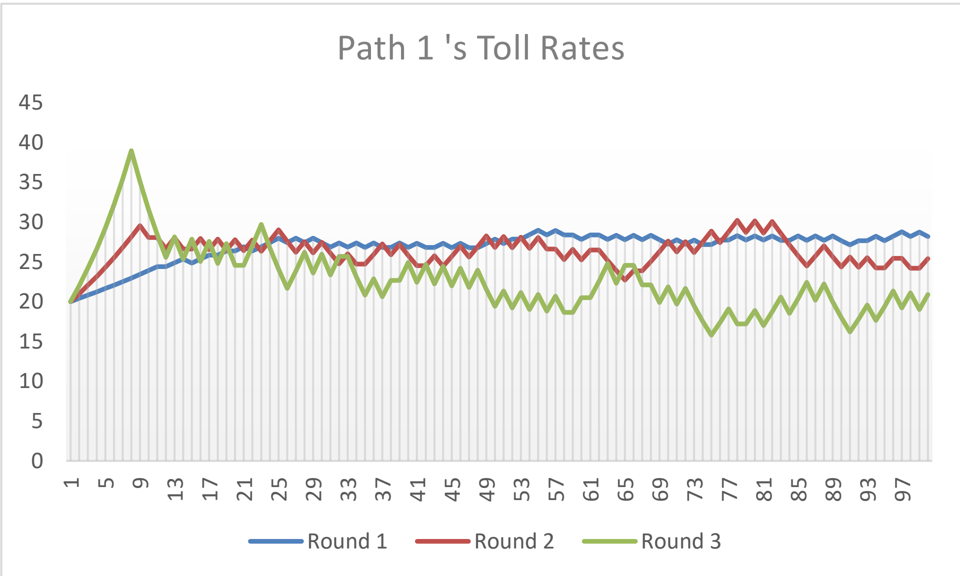

Figure 8. Line Chart - Path1 Toll rates

further investigate the trend of Path-1's toll rates in the extended 100 days, the converged values in three rounds distinct and are shown on the line graph above. The toll rates in three rounds converges to 27.98, 26.35 and 19.97 from Day 50 to

Day 99. The toll rates under equilibrium is actually decreasing as the daily increment ratio get larger from 0.02 to 0.10. A possible rationalization of this result could be that, as the player distribution converges to the equilibrium value – 0.5, there is still a fluctuating difference between the numbers of two paths. In the meantime, the program still regards the path with more players as the congested path, although the congestion is relatively small. As the relatively congested path shifts between the two paths, both path have their toll rates increase and decrease in order. Mathematically, since $(1 - 0.1) * (1 + 0.1) = 0.99 < 1$, the toll rates gradually drop as a result.

In summary, the decreasing trend of toll rates in equilibrium case could be attributed to the fact that the program is too sensitive to the tiny congestion. To overcome this issue, we modify the code, change the toll-increment setting to a new mechanism in which only consecutive-three days of being the relatively congested path will trigger the toll rate to increase. Vice versa. We made this modification, aiming to adjust the program to be inert to the tiny congestion. This modification turned out to work as

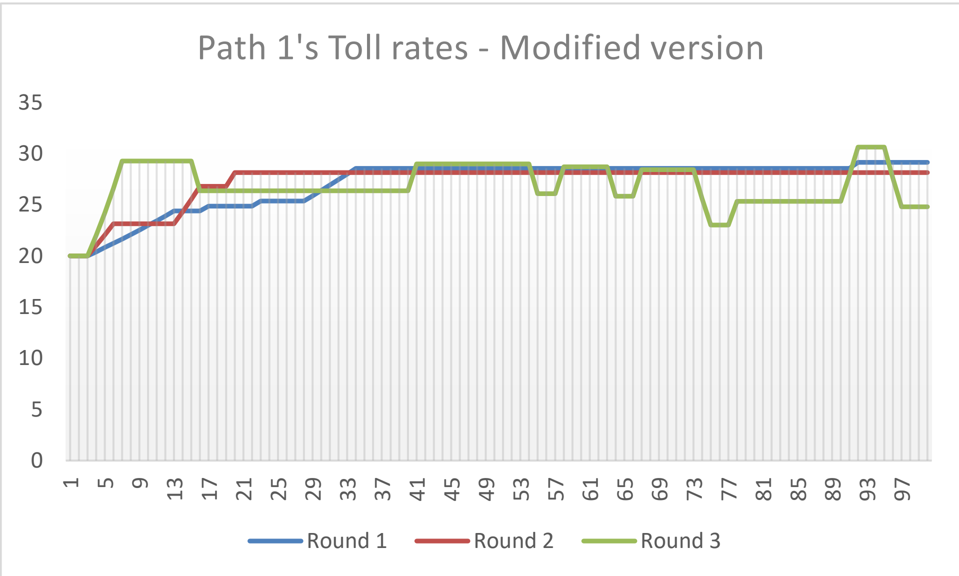

Figure 9. Line Chart - Toll rates - modified

updated toll rates of three rounds all converges to the same value around 27 and 28.

4.1.4 Model with upper/lower-bounded congestion-driven toll rates

In real-life scenario, toll rates of a tunnel could hardly be driven by congestion without any restriction. For the company operating the tunnel, the charging toll, as part of their revenue sources, are far more than a congestion-influencing factor. In this way, we would like to discuss the case in which toll rates are bounded.

To be noticed, we would assume that the ranges of two kinds of toll rates have no overlapping. This assumption is reasonable since the overlapping of ranges could result in the equilibrium in which two toll rates converges to the same value in the overlapping area, making this discussion meaningless. We also predict that, given two toll rates varying in ranges without overlapping, the final toll rates will also converge to the upper bound and the lower bound respectively, equivalent to the case discussed in section 4.1.2.

We control the (upper bound, lower bound) for the two paths to be [10, 20], [40, 50] respectively, and run the test in 100 days.

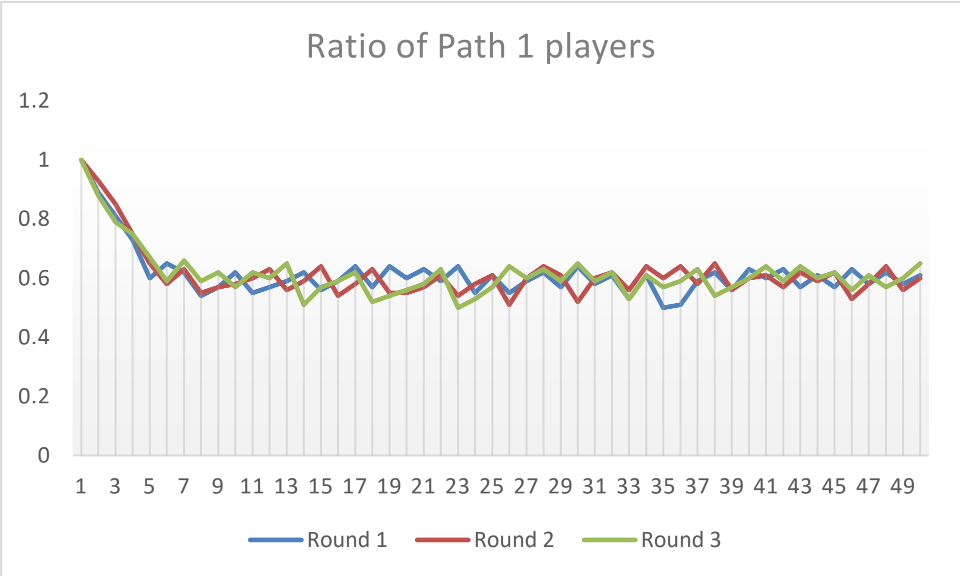

Figure 10. Line Chart - Bounded toll rates

The converged ratio of players choosing path-1 is unsurprising around 0.6, as is suggested in previous section. The converged and bounded toll rates for the two paths are 20 and 40 respectively. This provides evidence for our assertion that the toll rates will converge to their bounds where the two values are closest. And the equilibrium case would identical to the case with two bounds as fixed toll rates.

4.2 Model of variations in Congestion

In this section we will investigate the payoff function based on another form of mathematical representation of congestion. As is introduced in Chapter 3.3, we will substitute the current congestion formula with a new form:

$$\text{Congestion} = \text{Constant} * \frac{Sensitivity}{Maximum\ Utility - Number\ of\ Players}$$

For easy comparison with previous results, it is important for us to scale the old formula so that it has the same ranges as the new one. Specifically, in the old congestion formula $\text{Congestion} = \frac{\text{Number of player}}{Total\ Number\ of\ players}$, the congestion values ranges in [0, 1] as the number of players increases. We will have the new congestion formula and the modified old formula as,

$$\text{Congestion}_{\text{new}} = 50 * \frac{1}{150 - Number\ of\ Players}$$

$$\text{Congestion\_old} = \frac{2}{3} * \frac{Number\ of\ Players}{100} + \frac{1}{3}$$

so that both congestion ranges in $[\frac{1}{3}, 1]$. Based on this, we do two rounds with different payoff-functions and fixed toll rates. The results as follow:

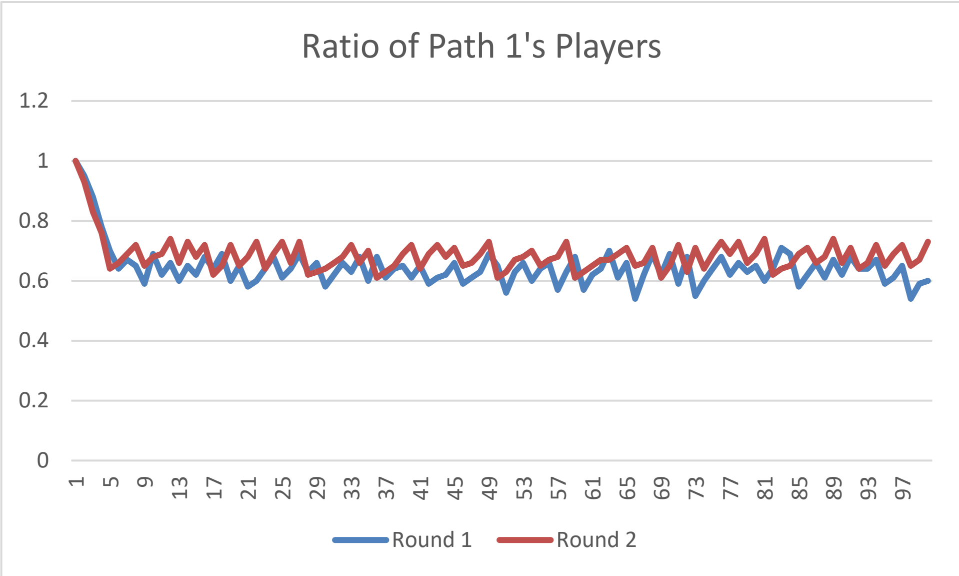

Figure 11. Line Chart - modified congestion

In two rounds, both ratios converge to a value around 0.7. However, by further calculating the average ratio during Day 50 to Day 99, we obtained two average ratios as 0.63 and 0.677. The mathematical explanation of the discrepance could hardly be given temporarily. However, despite the difference between two kinds of mathematical representation of congestion, both formulas are monotone in their domain of [0, 100]. Based on this, we could assert that, with this new version of congestion formula, the ratio would still converges compared with the scenarios we discussed in section 4.14. Although the exact equilibrium might shift a little, we are guaranteed that an equilibrium will still be reached.

Chapter 5 Difficulty and Limitation

5.1 Difficulty encountered

This project is mainly a theoretical analysis of the real-life problem under the specific scenario of Hong Kong harbor tunnels. The conceptualization of the congestion problem, as well as the program implementations are the main dish for the project.

The project concerns with the problem that has long been evaluated and studied by many researchers. I should admit that many solutions proposed by previous works seems to be theoretically reasonable and promising. Yet due to my limitation of ability and knowledge, I could hardly comprehend every possible solution proposed in the past. As a result, with the opinion proposed by my supervisor Dr. Huang, I tried my best in re-creating a congestion scenario, designing an appropriate toll-rate computing algorithm and testing its feasibility.

5.2 Limitation of the project

The implementation of the methodologies is mainly based on the hypothetical modeling of the real-time congestion problem. The derivation of payoff function and other formula includes the usage of several assumptions as foundation. The solution we proposed is also on full basis of a conceptualized model. Regarding the feasibility of the solution, we still need to verify the likelihood that our modeling could substantially represents a real-life congestion issue.

Chapter 6 Conclusion

This project provides a thorough analysis of hypothetical congestion problem modeled from the real congestion issue happened in Hong Kong harbor tunnels. It aims to propose a theoretically-proved solution, applying meticulously computed toll rates on the tunnels for the purpose to mitigate the congestion.

The project generalizes the congestion problem with a specific scenario of congestion game. It further discusses how the players would perform under different settings – with and without the imposed toll rates, whether the toll rates is fixed or dynamic, whether the toll rates are bounded. Based on the test result, it has been proposed that the congestion-driven toll rates applied to both paths could effectively shift the equilibrium to a relatively even state. The project implements the congestion-driven toll rates with fixed ratio of daily increment/decrement. It has been found out that a relatively smaller increment ratio could result in a stable equilibrium. While the larger increment ratio might incur instability, this issue could be solved by introducing a new inertia mechanism, and lowering the algorithm's sensitivity to congestion.

Further effort for this project could focus on the proving the feasibility of congestion-driven toll rate applied to actual congestion issue. Also, the quantification of congestion and toll rates relative to the driving experiences could be discussed and further evaluated.

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