# A Game-theoretic and Algorithmic Study of the Toll Rates of Hong Kong Road Tunnels

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# Abstract

Traffic networks in Hong Kong have been suffering from extensive congestion for years, especially areas near the three harbor tunnels connecting Kowloon and Hong Kong Island. A specific pattern of congestion emerges that one of the three tunnels, the Western Harbor Crossing, is usually free of congestion, due to a probable factor of its relatively high toll rates compared with the other two.

In this project, the specific congestion issue occurred at three harbor tunnels in Hong Kong and their peripheral areas will be focused. Modeling will be carried out in transforming the real congestion problem into congestion game discussed in Game Theory areas. Based on this, the algorithmic problem of computing the best toll rates, potentially in both the offline and online settings, will be further explored.

On the lower level, works will be carried out mainly on designing a network model that reasonably describes the actual features of traffic conditions around three harbor tunnels in Hong Kong, and effectively reflects their performance provided with drivers' behaviors. On the upper level, efforts will be invested on the design of algorithm that computes the toll rates in reducing traffic congestion theoretically. A demonstration is also expected in proving the utility of our modeling and algorithm as well.

At the time of writing, modeling of the congestion game is ongoing. Particularly the features of the tunnels' traffic network are being investigated.

## Acknowledgement

I want to present my greatest appreciation and gratitude to Dr. Huang Zhiyi for supervising and helping me with the entire project. This is the first project of theoretic computer science I have ever participated in, and Dr. Huang has been offering support and guidance in every aspect throughout. I also want to thank my CAES lecturer Cezar Cazan, who has been offering me advice on technical report writing and has been very helpful.

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## Abbreviations

| | |
|---|---|
| CHT | The Cross-Harbor Tunnel |
| EHC | The Eastern Harbor Crossing |
| WHC | The Western Harbor Crossing |
| PoA | Price of Anarchy |

# Chapter 1 Introduction

## 1.1 Problem specification

With the assistance of GPS system and audio broadcast offering real-time traffic information, congestion still seems an insurmountable urban issue prevailing in most metropolises around the world. Specifically, a pattern of congestion is noticeable that those bridges or tunnels that locates at crucial position in a city, connecting two major areas for example, are more likely under a constant congestion throughout daytime.[1]

Here in Hong Kong, three major tunnels in parallel connects Hong Kong Island to Kowloon. Specifically, the Cross-Harbor Tunnel and the Eastern Harbor Crossing are more likely suffered from congestion than the Western Harbor Crossing. From the perspective of theoretic computer science, this system has a relatively low efficiency as the capacity of WHC is not effectively used while CHT and EHC are exploited. This phenomenon could be ascribed to the relatively higher toll rates issued by WHC, twice as the rate issued by EHC and three times as the one issued by CHT. [7]

The difference in toll rates contributes to the uneven flow distribution among harbor tunnels. This motivates our project to model the real congestion issue as the congestion game in the field of Algorithmic Game Theory.[2][5] Specifically, we might draft a start-destination flow network based on geographical reality of harbor tunnels in Hong Kong, see the drivers as rational players and treat the toll rates as a controllable factor among all the factors that incurs a cost onto the players. By manipulating the toll rates, the modeling could potentially lead us to a theoretical solution of congestion, an impartial distribution of driving flows consequently.

### 1.2 Previous works

Previous researchers have made remarkable effort in exploring the relevant problems. Richard Cole and his colleagues developed a theory on the modeling of pricing network problems, which suggested a solution in the general form and proved its feasibility provided with certain conditions.[4] This result provides a theoretical base and motivates the feasibility of our project to a certain extent. Besides, Hai YAN and his colleague from civil engineering field discussed the possibility of applying road tolls as a general strategy in tackling queueing and congestion problems, and they also proposed a reasonable algorithm in computing the road tolls and proved the stability of their algorithm under some simple cases.[1][3] While these works have suggested from theoretical perspective the feasibility of controlling congestion traffic with toll rates, it should be noticed that few work was found in applying the relevant theories to a specific congestion problem with the modeling approach, probably as the consequence of the complexity incurred by modeling a real routing problem into a flow network with agents.[6]

### 1.3 Scope

This project focuses on the specific case of the congestion problem and uneven driving flow occurred among three harbor tunnels connecting Kowloon and Hong Kong Island. Especially the major effort will be devoted in the design and enhancement of the modeling of the flow network. The reason is two-fold. On the one hand, the three parallel tunnels together with their peripheral areas are of a geographically simple pattern compared with any specific road network of a city. As the tunnels stand out to be a relatively independent existence, it makes it easier to enhance the quality of our model without considering the triviality of traffic. On the other hand, the solution is expected to mitigate the congestion problem in harbor tunnels. By concentrating on the application of algorithmic game theory on reality, the model is anticipated to be refined as closed to the actual performance as possible. A feasible toll-rate computing method is also a desirable outcome of the project.

### 1.4 Outline

This report will unfold as follow. In chapter 2, the methodology of the project will be delivered, with detailed demonstrations of the approaches taken. Chapter 3 mainly discusses the current progress of the project. It also offers details about a model that was evaluated during the progress. And this is followed by the limitation of our project and the difficulties we have encountered in chapter 4. The interim conclusion will be made in chapter 5 at last.

# Chapter 2 Methodology

In this chapter the approaches of the project will be introduced and rationalized. The implementations for a concrete model will be discussed, followed by the expected cascading development of our model.

The project is mainly based on theoretical modeling of actual cases in Hong Kong harbor tunnels. In the introduction part it has already been shown that the variances of toll rates among different paths result in the uneven distribution of driving flows. This indicates a potential availability of controlling driving flows' distribution with meticulously calculated toll rates, and it motivates the approach, having toll rates as the major variable in the model. Based on the choice of major variable, and the illumination of work conducted by previous researches, the congestion game is an ideal candidate of the model. Before the specification of modeling, a few concepts and theorem will be introduced in advance.

## 2.1 Introduction of concepts and theorems

### 2.1.1 Definitions and Concepts

**Potential game [9]-** In game theory, a game is said to be a potential game if the incentive of all players to change their strategy can be expressed using a single global function called the potential function.

**Congestion game [9]-** Congestion game is a kind of potential game, in which we define players and resources. The payoff of each player depends on the resources which the player chooses, and the number of players choosing this same resource (in other words, how congested the resource is).

**Nash equilibrium [10]–** Nash equilibrium in game theory is a solution concept of a non-cooperative game involving two or more players. In Nash equilibrium, each player is assumed to know the equilibrium strategies of the other players, and no player benefits from changing his own strategy.

**Pure strategy and pure Nash equilibrium [10]-** Pure strategies are the strategies when every player chooses the same strategy each time. It contrasts with mixed strategies when player has a set of probabilities defined for his strategies. Pure Nash equilibrium is Nash equilibrium containing only pure strategies for every player. The opposite is mixed Nash equilibrium.

**Selfish routing [8]–** Selfish routing is a concept first introduced by Tim Roughgarden in his thesis in 2002. It defines, in a congestion game, a player's selfish, uncoordinated behavior.

**N**onatomic selfish routing – nonatomic selfish routing is a model of routing that assumes there are very large number of players, each controlling a negligible fraction of the overall traffic. This contrasts with atomic selfish routing, in which each player controls a nonnegligible amount of traffic.

**Price of Anarchy [8]–** Price of anarchy (PoA) is a concept in game theory that measures how efficiency of a system degrades due to selfish behavior of its agents. As far as the project concerns, the price of anarchy of a congestion game could be defined as follow:

$$\mathrm{POA} = \frac{total\ cost\ of\ routing\ in\ equilibrium\ case}{total\ cost\ of\ routing\ in\ optimal\ case}$$

**Pigou-like Network -** A Pigou-like network (see Figure 1) has:

- Two vertices, s and t.
- Two edges from s to t.
- A traffic rate $r > 0$.
- A cost function $c(\cdot)$ on the first edge.
- The cost function everywhere equal to $c(r)$ on the second edge.

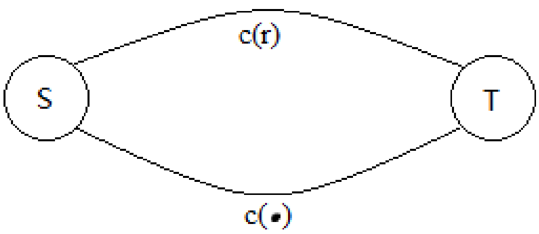


Figure 1 Pigou-like network

2.1.2 Theorems

**Theorem 1** Rosenthal's Theorem [4]

*Every atomic selfish routing game, with arbitrary real-valued cost functions, has at least one equilibrium flow.*

**Theorem 2** Tight POA Bounds for Selfish Routing [4]

*For every set $\mathcal{C}$ of cost functions and every selfish routing network with cost functions in $\mathcal{C}$, the POA is at most $\alpha(\mathcal{C})$ where:*

$$\alpha(\mathcal{C}) := sup_{c\in\mathcal{C}}\ sup_{r\geq 0}\ sup_{x\geq 0}\left\{\frac{r\cdot c(r)}{x\cdot c(x)+(r-x)\cdot c(r)}\right\}$$

## 2.2 Construction of model

This project models the congestion problem of Hong Kong harbor tunnels as a congestion game. Specifically, the three tunnels connecting Hong Kong Island and Kowloon, which are CHT, EHC, WHC respectively, are modeled as resources. Drivers passing the tunnels are modeled as the player who apply selfish routing as their behavior. As for the payoff function, in other way the cost function, of a resource, the toll rates and other relevant factors will be taken into consideration.

This kind of modeling is reasonable as it was applied in previous researches.[1] More importantly, it can relatively reflect the actual case of the congestion problem in the harbor tunnels while generalizing the real-life problem to the degree to be theoretically analyzed. A primary reason is that the three harbor tunnels being a relatively independent existence from their peripheral traffic are suitable to be considered as three individual paths in a network. And this fact extremely facilitates and simplifies the further calculation and computation, as it eliminates a common difficulty faced by a modeling of real-life traffic network, the complexity of geometrical frame of the network.

The reason of applying selfish routing as the model of drivers, the players in this game, is also straightforward. A normal driver would prefer a route with fewer toll charge to one with higher charges, and prefer a relatively fast, lower-time-consumed route to an opposite. This generalizes the selfish behavior of a real driver, which corresponds to the model of selfish routing. Moreover, a normal driver has little information about other drivers, and it is difficult, and even impossible for them to decide what influence their individual driving behavior would exert onto the entire traffic network, due to their little awareness of the entire, macro traffic. And this leads to a conclusion that no substantial coordinating behavior would exists among drivers in a traffic network. In this way, the selfish routing is ideal and most appropriate to describe the driving behavior of all the players in this game.

As for the payoff function, the toll rates applied to the tunnels, as well as other factors concerning the traffic and cost of the route, are generalized as cost functions incident to paths in the network of the model. Because the payoff of a driver passing a harbor tunnel is substantially negative, it can be summarized as a ‘cost’ to the driver. And this cost can be derived based on the consuming time of routing, the toll charge of routing, and possibly other factors relevant to the traffic. Based on this assumption, it is justified to apply toll charges and passing time as contributing factors to the payoff function of a route.

As the choices of the model and their rationale being presented above, what follows the construction of model is the further processing and computation. Provided with the choices of resources (the routes corresponding to tunnels), their payoff function, and the players' selfish routing model, the Nash equilibrium will be computed with the support of Rosenthal's Theorem mentioned in section 2.1.2. The Nash equilibrium, as is defined, is the solution concept of the game in which no player would benefit from changing his strategy. Considering that a driving flow is relatively in a large quantity of vehicles and the general assumption that individual influence could be negligible for a single driver in a driving flow, the Nash equilibrium would be a proper indication of players' behaviors in this congestion game. In other words, the result in the Nash equilibrium could reasonably represents the actual driving flow in a traffic network with strong theoretical base.

## 2.3 Processing model with the factor

As is indicated in the project title and the introduction, toll rate is the primary factor used in this project in controlling and mitigating congestion. And it is one of the major concern of this project that, before the toll rates are decided and applied, it should be primarily investigated how the toll change would influence the equilibrium of the game, the outcome of players' behavior. Mentioned in section 2.2, toll rate is modeled as a contributing factor to the payoff function. It would occur as a natural thought to compare the equilibria of the games with and without toll rate as a contributing factor.

In accordance with this logic, the processing of model would unfold in the following way. First the model without toll rate will be investigated and the equilibrium under this circumstance will be computed. Then the toll rate could be taken into consideration and generalized as a term in the payoff function. The corresponding equilibrium under the updated circumstance will be re-computed and compared with previous result. This step can also be expanded into several iterations by modifying the toll rate's algebraic term in payoff function into various mathematical forms. For example, the contribution $F(t)$ of toll rate $t$ to the player's payoff function can be evaluated as a constant,

$$F(t) = C$$

a linear function,

$$F(t) = at + b$$

a high-degree polynomial,

$$F(t) = \sum_{i=0}^{n} a_n \cdot t^n$$

and even more forms. The variation the toll rate's contribution in the payoff function is reasonable. Because the payoff function is primarily corresponding to the consuming time of the routing. To include the toll rate as a factor, a hypothetical comparison between the value of time and the value of toll rate is necessary. As there is no existing formula to define or decide the algebraic formula of toll rate, it would be our choice to make several attempts on different formula of toll rates' contribution and discuss and compare the result later.

## 2.4 Development and modification of model

With the model specification provided in section 2.2 and model processing in section 2.3, the fundamental frame of our modeling approach has already been presented. In following them, this section will mainly discuss how the model is expected to be improved and evolve at different stages of the project.

Basically, attempts have been (and will be) made in improving the model in two directions: the resource and the players.

As for the resource, the tunnels in other words, the major focus of the project is on the improvement of geometrical and geographical design of the network, making it close to the actual case as possible. A cascading illustration is given in Figure 2. The left graph presents the initial network with three parallel paths representing three tunnels. It is simple and ideal for computation, but substantially reflecting the real feature of Hong Kong harbor tunnels to a limited degree, as it assumes that three tunnels share the exact same starting and ending points, and that all the drivers are departing and completing their trip at same points. To improve from it, the model is evolved to the middle network, in which drivers have individual starts and destinations. Eventually the peripheral areas around the ending points of harbor tunnels will be taken into consideration. This choice is reasonable because in real-life cases, the congestion in the harbor tunnel is not only a consequence of huge driving flow inside the tunnel, but also a result of potential congestion near the starts and ends of the tunnel. And in this way the peripheral areas around the tunnels' endings are relevant and necessary to be theoretically investigated.

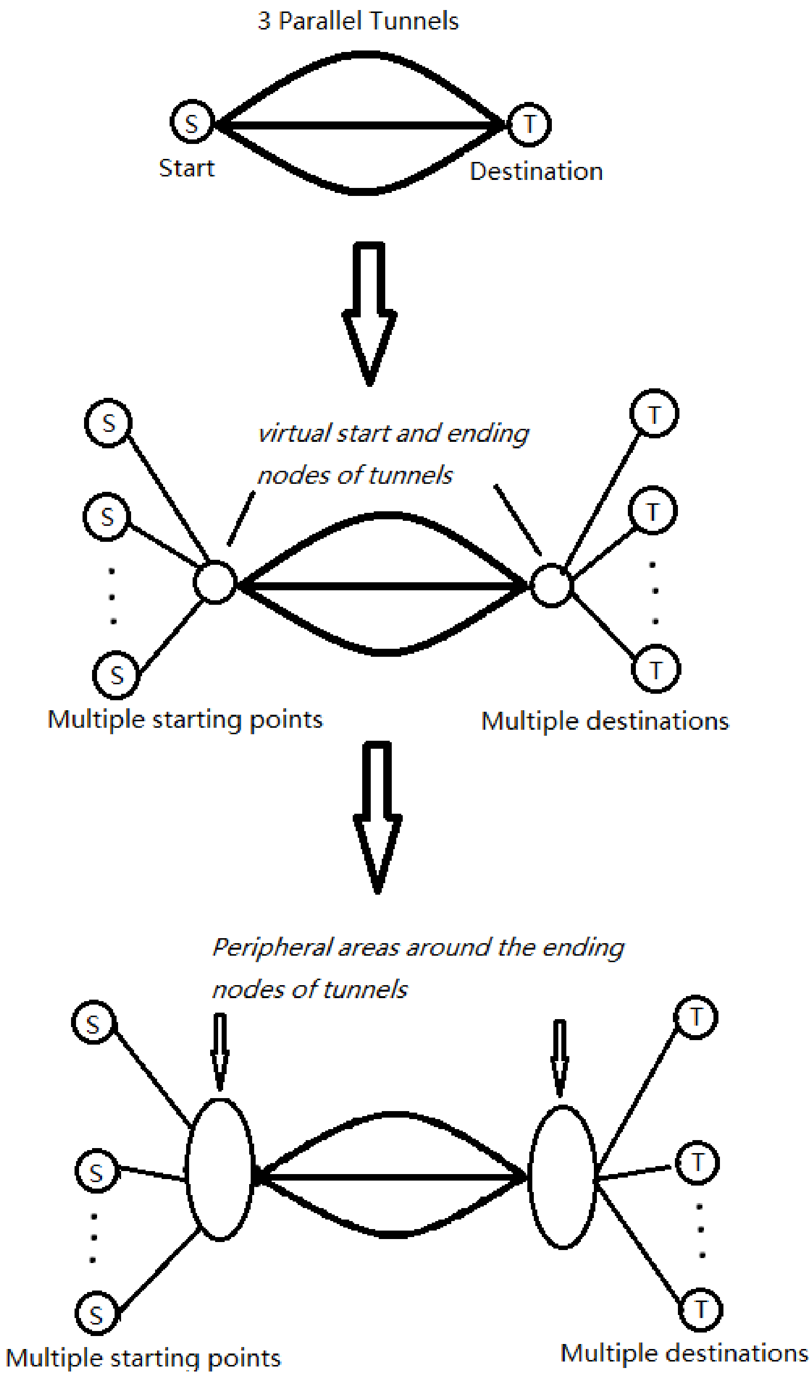


Figure 2 Models with different graphical design

Regarding the players in this congestion game, the project will start from offering the players a pure strategy. And a pure strategy applied indicates that the driver will choose the same resource, the same route, under the same circumstance. This is reasonable to certain extent, as drivers usually have some tendencies regarding their choice of route, and they probably make similar choice under the same situation. However, the randomness in drivers' choice-making is also noticeable. And a simple approach to generalize this kind of randomness in our model is to transform the pure strategies to the mixed strategies taken by players. A simple example could be that driver A has the tunnel CHT as his first preference and will choose this route with 80% probability, while in the rest 20% occasions he will choose the route based on expecting payoffs of routes. This kind of improvement in the players' modeling will incur much complexity in computation, yet it is meaningful as being representative of a real driver's behavior.

## 2.5 Summary of methodology

To conclude all the methods mentioned in this chapter, a list of modeling objectives, with regards to each factor the project is investigating, has been summarized as follow:

1 Toll rates

  1.1 Model without toll rates

  1.2 Model with fixed rates

  1.3 Model with rates in different algebraic formula

2 Design of network

- 2.1 A simplified single-source-single-destination graph
- 2.2 A many-source-many-destination graph
- 2.3 A many-source-many-destination graph while taking the peripheral areas of tunnels' endings into consideration

3 Players' behavioral pattern

- 3.1 All players applying pure strategy
- 3.2 Players applying mixed strategy

4 Factors that affect agents' choice

- 4.1 Estimated passing time (current situation of congestion in another way)
- 4.2 Drivers' own demand

# Chapter 3 Project Progress

This chapter is to reveal the current progress of the project

In accordance with the modeling objectives set in section 2.5, the project is following its path and currently the modeling objectives 1.2, 1.3 and 2.2 are ongoing. The change of equilibrium incurred by the introduction of various algebraic formula of toll rates' function is being investigated. The influence of replacing the single-source-single-destination graph with the many-source-many-destination graph is being evaluated. The pure strategies applied by drivers is still preserved.

Up to the moment, most of the calculations for the equilibria and the Price of Anarchy were taken on draft papers by hand. However, due to the limitation of the progress, and the fact that many results are not complete and waiting to be tested, only the simple models and calculating process will be presented in this interim report.

### 3.1 Model of variation of Pigou-like network

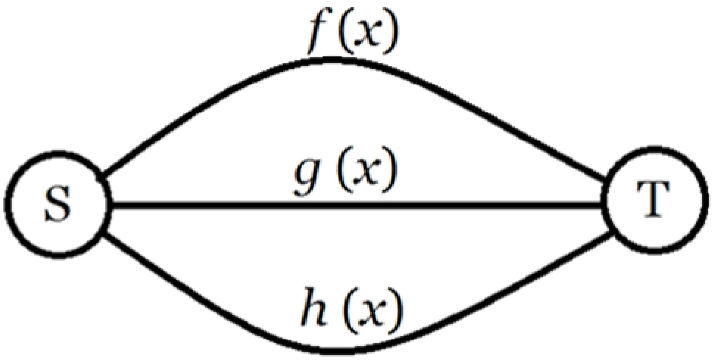


Figure 3 Variation of Pigou-like network

In this variation of Pigou-like network, we define the payoff functions as the cost of each route (which means that a lower cost is desirable). And define the three cost functions incident to tunnels as follow:

$$f(x) = 2,\ \ g(x) = 1,\ \ h(x) = x,\ \ 0 \leq x \leq 1$$

In which $x$ represents the flow on the resource, and the total flow is 1.

As we noticed that,

$$f(x) > g(x) \geq \ h(x)$$

Based on the definition of Nash equilibrium, all the drivers will flood to the bottom route. And in equilibrium the flow distribution among three tunnels is $(0, 0, 1)$. The total cost in equilibrium is

$$0\ \cdot f(0) + 0 \cdot g(0) + 1 \cdot h(1) = 1$$

In optimal case, some drivers, though reluctantly, are taking the middle route, while others taking the bottom route. No one takes upper route obviously as it is always worse than the middle. Assume flow $x$ takes the middle route. Then in optimal case, the total cost is

$$x \cdot g(x) + (1 - x) \cdot h(1 - x)$$

$$= x + (1 - x) * (1 - x)$$

$$= x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$$

Then the Price of Anarchy of this game is

$$POA = \frac{4}{3}$$

### 3.2 Enhancement of model in 3.1

For this model, assume everything are the same as the ones in model 3.1, except that the cost function $h(x)$ changes to

$$h(x) = x^p, p \to \infty$$

Then in the equilibrium case the total cost will remain 1. However, the cost of optimal case shifts

$$\epsilon \cdot g(\epsilon) + (1 - \epsilon) \cdot h(1 - \epsilon)$$

$$= \epsilon + (1 - \epsilon) \cdot (1 - \epsilon)^p = \epsilon + (1 - \epsilon)^{p+1}$$

Then the total cost approaches 0 as $\epsilon \to 0$.

The Price of Anarchy in this case will be $\infty$.

This extreme case shows that the Price of Anarchy is unbounded if the cost function increases its degree. A high value of PoA represents the inefficiency of the system and is not desirable. And this stimulates a temporary conclusion that the high degree of payoff function should be avoided. And the corresponding toll rates will be evaluated to check the validity of this conclusion.

# Chapter 4 Difficulty and Limitation

### 4.1 Difficulty encountered

This project is mainly a theoretical analysis of the real-life problem under the specific scenario of Hong Kong harbor tunnels. The deductions of mathematical equality and inequality are the main dish for the project. Inevitably, some theoretical relationships are hard to deduct or calculate.

Moreover, apart from the Price of Anarchy being as an index for the evaluation of our model, more indices are expected to reflect the effectiveness of a particular model. The Price of Anarchy as a single index of efficiency, is limited in evaluating the payoff functions of the model. And cases occurred during the calculation in which the PoA of the model seems promising, while the flow is proved to flood to a single resource and result in the congestion.

### 4.2 Limitation of the project

The analysis carried out in this project is on full base of the actual conditions in Hong Kong harbor tunnels, and aims to contribute to the congestion issue. The specification of the scenario which the project applies implies a potential loss of generality in evaluating and solving the congestion problem as a general issue faced by most metropolises.

Lack of proficiency in mathematical problem solving is also a limitation of the project. We have proposed various ways and possible directions to analyze the congestion game model. Yet it might be a problem that not all of the anticipated approaches will be carried out and accomplished.

# Chapter 5 Conclusion

This project provides a thorough analysis of congestion issue in Hong Kong harbor tunnels within the concerned fields of algorithm and game theory. It aims to propose a theoretically-proved solution, applying meticulously computed toll rates on the tunnels for the purpose to mitigate the congestion.

The project takes the approach of modeling the issue as a congestion game. By exerting a proper set of toll rates based on current flow distribution in the network, the equilibrium is expected to shift to a desirable outcome.

At the time of writing, the modeling of this project is ongoing. Efforts have been made in exhausting the potential factors of congestion, improving the complexity of the model's network and exploring the possibilities of desirable toll rates' functions.

Based on the current progress of the project and the difficulties that have been encountered, future work will concentrate on three issues:

1. Improving the mathematical problem-solving skills.
2. Optimizing and completing the evaluation of a model
3. Continuing with model development and computation

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