COMP2121A: Discrete Mathematics
Tutorial 2 13 Sep, 2016

The last two questions are extra questions for those who want more practice, and we might not have time to cover them during the tutorial session.

1. Define $f(n) = 1^2 + 2^2 + 3^2 + \ldots + n^2$. Use mathematical induction to prove that $f(n) = \frac{n*(n+1)*(2n+1)}{6}$ for all positive integers.

2. Given a finite set A of n points on the plane (2-dimensional space) such that for any two points x, y in A, the line containing x and y must contain another point z in A. Prove that all points in A are on the same line.[1]

   (a) Is the following proof (induction on the number of points) correct? If not, where is the bug?

   - Base case: for point set of size 3 the statement is true.
   - Inductive step: assume this statement is true for point set of size $k \geq 3$. Consider the case when we have a point set A of size $k + 1$. We argue as follows.
     i. Pick A' of k points from the given point set A. Let x be the other point in A but not in A'.
     ii. By *induction hypothesis*, points in A' are on the same line.
     iii. Pick any y in A', the line going through x, y contains another point z in A.
     iv. Thus, x, y and z are on the same line.
     v. So x and all points in A' are on the same line.

   (b) Can you give a proof by contradiction?

3. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

4. Let p denotes the statement "$\sqrt{2} + \sqrt{3}$ is an irrational number", q denotes the statement "$\sqrt{6}$ is an irrational number".

   (a) Prove that q is true. (That is, $\sqrt{6}$ is irrational.)

   (b) Prove $\neg p \rightarrow \neg q$ is true. (That is, if $\sqrt{2} + \sqrt{3}$ is rational, then $\sqrt{6}$ is rational.)

   (c) Is $\sqrt{2}+\sqrt{3}$ rational or irrational? Prove it using logical arguments and the results from (4a) and (4b).

[1] This is equivalent to Sylvester-Gallai theorem, which is named after James Joseph Sylvester, who posed it as a problem in 1893, and Tibor Gallai, who published one of the first proofs of this theorem in 1944.