COMP2121A: Discrete Mathematics
Tutorial 6 1 Nov, 2016

Question 1: Random Relation (20 points)

Suppose A is a set of size n, where n is a positive integer. Recall that a relation R on A is a subset of $A \times A$. If an element x is related to an element y by R, we write $(x, y) \in R$.

We now form a random relation R in the following way. Suppose p is a real number in $[0, 1]$. *Independently* for each $(x, y) \in A \times A$, the event "$(x, y) \in R$" occurs with probability p.

For each of the following statements, do the following.

(1) Translate into an equivalent logical statement with quantifiers.

(2) Calculate the probability with which the statement holds. If your answer is wrong, partial credit is given only if you show your working.

Example Statement. The relation R is empty.

(1) $\forall x \in A, \forall y \in A, (x, y) \notin R$.

(2) Since each pair (x, y) is not included in R with probability $1 - p$ independently, the probability that the statement holds is $(1 - p)^{n^2}$.

(a) **(4 pt)*** The relation R is reflexive.

(b) **(4 pt)*** The relation R is symmetric.

(c) **(4 pt)*** The relation R is anti-symmetric, i.e., for all $a, b \in A$, if both (a, b) and (b, a) are contained in R, then $a = b$.

(d) (2 pt) The relation R is symmetric and anti-symmetric.

(e) (3 pt) The relation R corresponds to a function from A to itself. For instance, if $(x, y) \in R$, this means that when the input is x, the output is y.

(f) (3 pt) The relation R corresponds to an injective function from A to itself.

(g) **Extra Credit. (10 points)**

The relation R corresponds to a bijective function from A to itself such that given any input, the output is not the same as the input.

Question 4: Dividing m Identical Coins Among n Persons (20 points)

Suppose $n \geq 2$ and m are positive integers. There are m identical coins to be distributed among n persons.

(a) We describe a procedure called UNFAIR to divide the coins. The first person comes along and an integer x is picked uniformly at random from $\{0, 1, 2, \ldots, m\}$. Then, the first person takes x coins and goes home. In general, when the ith person comes along (where $i < n$), and there are r coins left, an integer y is picked uniformly at random from $\{0, 1, 2, \ldots, r\}$ and the ith person goes home with y coins. The nth (last) person just takes whatever that is left.

Define X_i to be the number of coins the ith person takes.

(i) **(4 pt)*** Compute $E[X_1]$, the expected number of coins the first person receives.

(ii) **(2 pt)*** Suppose $n \geq 3$. Given that the first person receives x coins, what is the expected number of coins the second person receives? (Compute $E[X_2|X_1 = x]$.)

(iii) **(2 pt)*** Assume $n \geq 3$. Compute $E[X_2]$.

(iv) (4 pt) For general $i \leq n$, compute $E[X_i]$.

(b) We next consider another procedure called FAIR. First, compute the set S of all integer solutions to the equation $x_1 + x_2 + x_3 + \ldots + x_n = m$, where each $x_i \geq 0$. A solution $(x_1, x_2, \ldots, x_n)$ is picked uniformly at random from S, and for each $1 \leq i \leq n$, the ith person receives x_i coins.

(i) **(2 pt)*** What is the size of S?

(ii) **(4 pt)*** Suppose X_1 is the number of coins received by the first person. What is the probability that $X_1 = k$, where $0 \leq k \leq m$? Express your answer in terms of n, m and k.

(iii) (2 pt) Prove that for all positive integers $n \geq 2$ and $m \geq 1$,

$$\sum_{k=0}^{m} k \cdot \binom{m+n-k-2}{n-2} = \frac{m}{n} \cdot \binom{m+n-1}{m}.$$