COMP2121A: Discrete Mathematics
Tutorial 7 8 Nov, 2016

Question 1: Random Inputs and Functions (15 points)

Let $n \geq 2$ be an **even number**. For $1 \leq i \leq n$, x_i is an independent input taking values in $\{0, 1\}$ uniformly at random, i.e., with probability $\frac{1}{2}$, x_i equals 1.

A random function $F : \{0, 1\}^n \to \{0, 1\}$ is picked from the following choices, each with probability $\frac{1}{4}$.

(1) $\mathsf{AND}(x_1, x_2, \ldots, x_n)$ equals 1 *iff* all x_i's are 1.

(2) $\mathsf{OR}(x_1, x_2, \ldots, x_n)$ equals 1 *iff* at least one x_i equals 1.

(3) $\mathsf{XOR}(x_1, x_2, \ldots, x_n)$ equals 1 *iff* an odd number of the x_i's equal 1.

(4) $\mathsf{MAJ}(x_1, x_2, \ldots, x_n)$ equals 1 *iff* **more than half** of the x_i's equal 1. (If exactly half of the x_i's are 1, the output is still 0.)

The random function F is applied to the random inputs to produce a random output $Y = F(x_1, x_2, \ldots, x_n)$.

1. (10 pt) For each of the 4 function choices, compute the conditional probability that the output is 1, given that particular choice for F. In other words, compute $\Pr[Y = 1 | F = \mathsf{AND}]$, $\Pr[Y = 1 | F = \mathsf{OR}]$, $\Pr[Y = 1 | F = \mathsf{XOR}]$ and $\Pr[Y = 1 | F = \mathsf{MAJ}]$. Express your answer in terms of n.

2. (5 pt) For the special case $n = 4$, compute the conditional probability that the function F is MAJ given that the output Y is 1.

Question 2: Hunters and Rabbits (10 points)

Suppose there are m different hunters and n different rabbits. Each hunter selects a rabbit uniformly at random independently as a target. Suppose all the hunters shoot at their chosen targets at the same time and every hunter hits his target.

(a) (2 pt) Consider a particular Rabbit 1. What is the probability that Rabbit 1 survives?

(b) (2 pt) Suppose $m = 7$ and $n = 5$. What is the expected number of surviving rabbits? Compute the answer up to 6 decimal places.

(c) (6 pt) Suppose $m = 7$ and $n = 5$. What is the probability that no rabbit survives? Compute the answer up to 5 decimal places.

Question 3: Moments of Binomial Distribution

Let $B = \mathsf{Bin}(n, p)$, i.e., flipping n biased coins, each having heads with probability p. Compute $E[B^2]$.

(For general $k \geq 2$, how to compute $E[B^k]$?)