COMP2121A: Discrete Mathematics
Tutorial 8 15 Nov, 2016

Question 1: Cycle Graph (10 points)

Prove that for $n \geq 3$, if a simple undirected graph is connected and each of its vertices has degree 2, then it must be the cycle graph C_n.

Question 2: A Party with n Persons (12 points)

Suppose there is a party with n persons. To avoid stating the obvious, when we say person x *knows* person y, we assume implicitly that x and y are different persons. Moreover, it is possible that person x knows person y, but person y does not know person x.

Suppose it is the case that for any two different persons x and y, the number of persons x knows is different from that for y, and the number of persons who know x is also different from that for y.

(i) (6 pt) Prove that for every person x in the party, the number of persons x knows plus the number of persons knowing x equals $n - 1$. (Hint: First prove that the least popular person (known by no one) actually knows everyone.)

(ii) (6 pt) Suppose a ball is being passed around. In each step, the person who currently has the ball passes it to a random person that he knows, i.e., if he knows r persons, he picks each one of them with probability $\frac{1}{r}$. Suppose the ball starts at the least popular person (who knows everyone else). What is the expected number of steps for the ball to reach the most popular person (who knows nobody else)? (You may express your answer in terms of a sum involving n.)

Question 3: Euler Circuits (15 points)

We consider undirected multi-graphs with no self-loops.

(a) (4 pt) Give necessary and sufficient conditions for an Euler circuit to exist in a graph.

(b) (3pt) Suppose a graph is connected and there are exactly 8 vertices with odd degrees. What is the minimum number of edges that need to be added such that the graph has an Euler circuit?

(c) (3pt) Suppose a graph has 4 connected components and all vertices have even degrees. What is the minimum number of edges that need to be added such that the graph has an Euler circuit that visits every vertex?

(d) (5pt) In general, given a graph, describe a method to add the minimum number of edges such that the graph has an Euler circuit that visits every vertex. Illustrate your method with a graph which has 10 connected components, and the number of vertices with odd degrees in each component is as follows: 0,2,2,4,6,8,8,10,10,12. What is the total number of edges added?