COMP2121A: Discrete Mathematics
Homework 1 **Due Date:** 7pm, 30 Sep, 2016

Rules: Discussion of the problems is permitted, but writing the assignment together is not (i.e. you are not allowed to see the actual pages of another student).

Course Outcomes

- **[O1. Abstract Concepts]**
- **[O2. Proof Techniques]**
- **[O3. Basic Analysis Techniques]**

1. (8 pt) **[O1]** Show that the following premises are inconsistent.
   (a) If Jack misses many classes through illness, then he fails high school.
   (b) If Jack fails high school, then he is uneducated.
   (c) If Jack reads a lot of books, then he is not uneducated.
   (d) Jack misses many classes through illness and reads a lot of books.

2. (12 pt) **[O2]** Show that $(p \to \neg q) \lor r$ and $p \to (q \to r)$ are logically equivalent.
   (a) using truth tables.
   (b) using logical equivalence.

3. (15 pt) **[O1]** Consider the following mathematical statement in number theory:

   For every integer n bigger than 1, there is a prime strictly between n and $2n$.

   (a) Express the statement in terms of quantifiers, variable(s), inequality symbols $<$ or $>$, logical operators $(\land, \lor, \to)$ and predicate $P(n)$: n is a prime number.
   (b) Express the negation of (a) without using the logical operator $\neg$.

   [Be careful to define the domain(s) of your variable(s)]

4. (10 pt) **[O1, O2, O3]** Prove the following statement. There exist irrational numbers x and y such that x^y is rational. (Hint: Consider $\sqrt{2}^{\sqrt{2}}$. Is it rational or not?)

5. (20 pt) **[O2, O3]** Use mathematical induction to prove following.
   (a) Prove that for integer $n \geq 60$, there are non-negative integers a and b such that $7a + 11b = n$.
   (b) Prove that 7 divides $3^{2n-1} + 2^{4n+1}$ whenever n is a positive integer.

6. (15 pt) **[O2, O3]** Prove that $f_n \mid f_{2n}$ (i.e., f_{2n} is divisible by f_n) for $n \geq 1$, where f_n is the nth Fibonacci number ($f_0 = 0, f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$ for $n > 2$). Hint: Use induction to show that $f_{2n} = f_n(f_{n+1} + f_{n-1})$, and you will need $f_{2n-1} = (f_n)^2 + (f_{n-1})^2$.

7. (15 pt) **[O1, O2]** Given a set S, a *partition* P of S is a collection of non-empty subsets of S such that every element $x \in S$ is in exactly one subset in P. (In particular, this means the subsets in P are pairwise disjoint.)

   Show that the notions of equivalence relation and partition are essentially equivalent via the following two steps.

   (a) Given an equivalence relation R on S, show that the collection of the equivalence classes of R forms a partition of S.

   (b) Given a partition P of S, define an equivalence relation R such that its equivalence classes form a partition of S.

8. (5 pt) **[O1, O2]** Decide whether the following statements about big-O notation are true or not.

   (a) Let $f(n) = n^2 + 2n$, then $f(n) = O(n^3)$.

   (b) Let $f(n) = \log n + 5$, then $f(n) = \Omega(\sqrt{n})$.

   (c) Let $f(n) = \sqrt{n} + 3 \log n + 5$, then $f(n) = \Theta(\sqrt{n})$.

   (d) Let $f(n) = n$, then $f(n) = O(\log^2 n)$.

   (e) Let $f(n) = 2n^2 + 5n + 4$, then $f(n) = O(n^2)$.