COMP2121A: Discrete Mathematics
Homework 2 **Due Date:** 7pm, 18 Nov, 2016

Rules: Discussion of the problems is permitted, but writing the assignment together is not (i.e. you are not allowed to see the actual pages of another student).

Course Outcomes

- **[O1. Abstract Concepts]**
- **[O2. Proof Techniques]**
- **[O3. Basic Analysis Techniques]**

1. (16 points) **[O3]** How many positive integers less than 10000
   (a) are divisible by 12?
   (b) are divisible by both 12 and 8?
   (c) are divisible by 12 but not by 8?
   (d) are divisible by either 12 or 8?
   (e) are divisible by exactly one of 12 and 8?
   (f) are divisible by neither 12 nor 8?
   (g) are palindrome? A number is a palindrome (written in base 10) if it is the same written backwards, e.g., 101, 33, 4, but 42 is not a palindrome
   (h) are palindrome and are even?

2. (12 points) **[O1]** How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 32,$$

   where each x_i for $i = 1, 2, 3, 4, 5$ is a non-negative integer
   (a) if $x_i \geq 5$ for $i = 1, 2, 3, 4, 5$?
   (b) if $x_1 \leq 5$ and $x_2 \geq 7$?
   (c) if $2 \leq x_1 \leq 8$ and $1 \leq x_2 \leq 6$?
   (d) if $1 \leq x_1 \leq 4$ and $3 \leq x_2 \leq 5$ and $4 \leq x_3 \leq 8$?

3. (10 points + 5 extra points) **[O2,O3]** An n-bit string is a string that consists of n symbols taken from $\{0, 1\}$. Let $p(n)$ be the number of n-bit strings that contain no occurrence of "000".
   (a) (4 points) Find $p(1)$, $p(2)$, $p(3)$ and $p(4)$.
   (b) (6 points) Find a recurrence relation for $p(n)$ with $n \geq 4$. That is, find an expression for $p(n)$ in terms of $p(n-1), p(n-2), \ldots$.
   (Hint: Consider all cases for the n-bit strings with no occurrence of "000". If the first bit is 1, then the last $n-1$ bits must not contain the string "000". What if the first bit is 0?)

(c) (5 extra points) Suppose n is your university ID. What are the last eight digits of $p(n^2)$?

(Hint: You can give the answer without explanation. Avoid overflow if you write a program for calculation. Suppose your university ID is $n = 3030000000$, then $p(n^2) \mod 10^8 = 95671809$.)

4. (10 points) **[O2]** Prove that any set of $n+1$ numbers taken from $\{1, 2, \ldots, 2n\}$ contains a pair a, b such that $a \neq b$ and a divides b.

5. (10 points) **[O3]** Three different factories X, Y and Z provide a certain automobile part. The probability that a part from X is defective is 0.12, the probability that a part from Y is defective is 0.08, and the probability that a part from Z is defective is 0.15. In a supply of 100 parts, 20 parts were obtained from X, 45 from Y and 35 from Z.

   (a) What is the probability that a part chosen at random from the 100 parts is defective?

   (b) If a chosen part is defective, what is the probability that this part is from X?

6. (12 points) **[O2,O3]** Suppose $S = \{1, 2, \ldots, n\}$ is the set of the first n positive integers. We construct a *random* subset R of S in the following manner. For each integer $k \in S$, toss a fair coin: if a head appears, then add k into R; if a tail appears, then do not add k into R. Therefore, each integer in S is added into R with probability $\frac{1}{2}$. For each $k \in S$, let I_k be a random variable such that I_k equals 1 if k is added to R and equals 0 otherwise. Let $w(R)$ be the sum of the integers in R.

   (a) What is the sample space for R? In other words, what is the set of all possible outcomes of R?

   (b) What is $w(R)$ in terms of $I_1, I_2, \ldots, I_n$?

   (c) What is the expected value $\mathbf{E}[I_k]$ for each $k \in S$?

   (d) What is the expected value $\mathbf{E}[w(R)]$ in terms of n?

7. (30 points + 5 extra points) **[O3]** In this question we consider the following puzzle.

   The director of a prison offers 42 death row prisoners, who are numbered from 1 to 42, a last chance. A room contains a row of 42 drawers. The director randomly puts one prisoner's number in each closed drawer. The prisoners enter the room, one after another. Each prisoner may open and look into 21 drawers, one by one, in any order. The drawers are closed again afterwards. If, during this search, every prisoner finds his number in one of the drawers, all prisoners are pardoned. If just one prisoner does not find his number, all prisoners die. Before the first prisoner enters the room, the prisoners may discuss strategy but may not communicate once the first prisoner enters to look in the drawers. Can you suggest a good strategy?

(a) (5 points) **First Attempt.** If every prisoner selects 21 drawers at random (independent from other prisoners), what is the probability that they will be released?

(b) (5 points) **Second Attempt.** Let's label the drawers from 1 to 42, from left to right. Let's consider the following strategy, prisoners numbered from 1 to 21 open the drawers from 1 to 21 and prisoners numbered from 22 to 42 open the drawers from 22 to 42. What is the probability that they will be released?

(c) (5 points) **Third Attempt.** Consider the following strategy. (We assume the drawers are numbered as in the previous question.)

i. Each prisoner first opens the drawer with his own number.

ii. If this drawer contains his number he is done and was successful.

iii. Otherwise, the drawer contains the number of another prisoner and he next opens the drawer with this number.

iv. The prisoner repeats steps 2 and 3 until he finds his own number or has opened 21 drawers.

Consider the following cases of 7 prisoners (each prisoner is allowed to open 3 drawers), if they use this attempt, will they be released?

Case 1

drawer ID	1	2	3	4	5	6	7	8
content	7	4	6	8	1	3	5	2

Case 2

drawer ID	1	2	3	4	5	6	7	8
content	3	1	7	5	8	6	4	2

(d) (5 points) To analyze the success probability of the Third Attempt, we can use the concept of permutation and cycles. A permutation σ on a set A is a one-to-one and onto function on A. Given a permutation, if we start with an element and repeatedly following the mapping (similar to Third Attempt) we will find a cycle. Actually a permutation can be represented by disjoint cycles. For example in our previous example, Case 1 can be represented in the following 3 cycles.

$$(1, 7, 5)(2, 4, 8)(3, 6)$$

Given a set A of n elements, if permutations on A are selected uniformly at random, what is the probability that this permutation has only one cycle?

(e) (5 points) Based on the previous question, let's consider our puzzle with 42 prisoners. The allocation of prisoner's number in drawers can be represented by a permutation. We know this permutation is selected uniformly at random, what is the probability that this permutation has a cycle of length l, for $l > \frac{n}{2}$.

(f) (5 points) If all prisoners follow the Third Attempt, what is the probability that they will be released?

(g) (5 extra points) Our previous analysis depends on the fact that the director of the prison put the numbers in the drawers randomly. The director may not want to help the prisoners. He can choose how to put the numbers in the drawers adversarially. In this case, do the prisoners still have a good strategy?