COMP2121A: Discrete Mathematics
Homework 3 **Due Date:** 7pm, 2 Dec, 2016

Rules: Discussion of the problems is permitted, but writing the assignment together is not (i.e. you are not allowed to see the actual pages of another student).

Course Outcomes

- **[O1. Abstract Concepts]**
- **[O2. Proof Techniques]**
- **[O3. Basic Analysis Techniques]**

1. (6 pt) **[O1]** How many vertices and how many edges does each of these graphs have?
   (a) Complete graph K_n
   (b) Cycle graph C_n
   (c) Complete bipartite graph $K_{n,m}$

2. (9 pt) **[O1]** For which values of $n \geq 3$ do these graphs have an Euler circuit?
   (a) Complete graph K_n
   (b) Cycle graph C_n
   (c) Complete bipartite graph $K_{n,n}$

3. (10 pt) **[O2]** Show that a simple graph G with n vertices is connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

4. (20 pt) **[O2, O3]** Let G be a simple digraph on n points with all indegrees and outdegrees at least $\frac{n}{2}$. The goal is to show that G contains a Hamiltonian cycle.

   Given a vertex x, its in-neighbours are $N^-(x) := \{a \in V : (a, x) \in E\}$ and its indegree is $|N^-(x)|$; its out-neighbours are $N^+(x) := \{b \in V : (x, b) \in E\}$ and its outdegree is $|N^+(x)|$.

   (a) Prove that in this graph G, we can find a cycle of length greater than $\frac{n}{2}$. (Hint. Try to extend a path as much as possible.)
   (b) Let cycle C be a maximum cycle in G. With the result of (4a), we know that $|C| > n/2$. If $|C| = n$, then we have found a Hamilton cycle, so from now on we assume $|C| < n$. Let $P = u_0u_1u_2 \dots u_k$ be a maximal path in the graph $G - V(C)$, by extending both sides as much as possible. Notice that the number of vertices in P is $k+1$. Define $S := N^-(u_0) \cap C$ and $T := N^+(u_k) \cap C$, prove that $|S|, |T| \geq n/2 - k$.
   (c) Let the distance $\text{dist}_C(x, y)$ be the number of edges from vertex x to y on the directed cycle C following its directed edges. Let $s \in S$, $t \in T$ be two different vertices on cycle C. Recall that C is a maximum cycle. Prove that $\text{dist}_C(s, t) > k + 1$.

(d) Given $s \in S$, define $F_s := \{v \in C : 1 \leq \text{dist}_C(s, v) \leq k+1\}$. From (4c), we know $F_s \cap T = \emptyset$. show that $|\cup_{s\in S} F_s| \geq |S| + k$.

(e) Finish the proof by combining previous steps that lead to a contradiction.

5. (55 pt) **[O2, O3]** Graph Coloring.

(a) (5 points) Prove that if G is a connected planar simple graph, then G has a vertex of degree at most five.

(b) (10 points) Prove that every connected planar simple graph can be colored using six or fewer colors.

(Hint. Use mathematical induction on the number of vertices. Apply (a) to find a vertex v with $deg(v) \leq 5$. Consider the subgraph obtained by deleting v and the edges incident to v.)

(c) (10 points) Prove that every connected planar simple graph can be colored using five or fewer colors.

You may prove by induction on the number of vertices.

i. Let G be a planar simple graph, and let v be the vertex whose degree is at most 5 (according to 5a). By induction hypothesis, $G - \{v\}$ can be five-colored. If v's neighbours are colored with less than 5 colors, then we can assign one free color to v. Otherwise, let the neighbours of v, when considered in clockwise order around v, be v_1, v_2, v_3, v_4, v_5 and their colors are $1, 2, 3, 4, 5$ respectively. Consider the subgraph with colors $1, 3$ (that is the vertices in G with colors 1 and 3, together with all edges between them). If v_1, v_3 are in different connected components in this subgraph, argue that we can switch their colors to get one color free for v.

ii. Continue with previous construction, if v_1, v_3 are in the same connected component in the subgraph, argue that we can switch the color of a vertex in $\{v_2, v_4, v_5\}$ such that there is one free color for v.

(d) (10 points) Given a set of lines in the plane with no three meeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that this graph can be colored with at most three colors.

You may follow the following steps.

i. Each vertex has at most 4 neighbours.

ii. Find a coloring order such that when a vertex is colored at most two of its neighbours are already colored.

(e) (10 points) We draw some circles on the plane (say, n in number). These divide the plane into a number of regions. Figure 1 shows such a set of circles, and also an alternating coloring of the regions with two colors. Now our question is: can we always color these regions this way?

You may follow the following steps to prove that the graph G representing regions is bipartite (hence can be two-colored).

i. Encode each region by $0, 1$ strings such that the code of adjacent regions differ by exactly one bit.

ii. Argue that any cycle on the graph G is of even length.

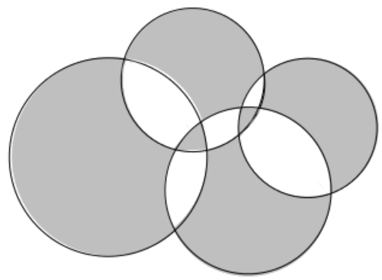

Figure 1: Two-coloring the regions formed by a set of circles.

(f) (10 points) If a planar graph has all degrees even, prove that the faces can be colored with two colors in such a way that any two faces with a common edge on their boundary get different colors.