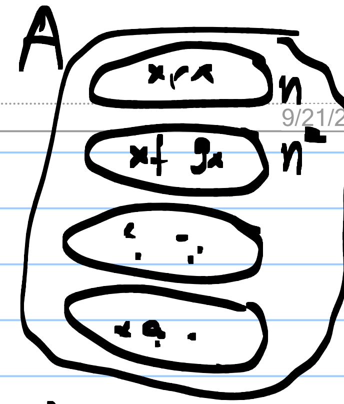
Note Title 9/21/2018

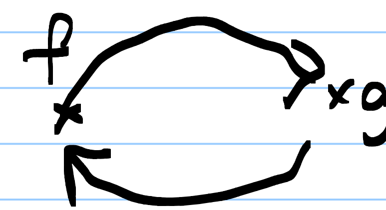
Asymptotic Notation and Equivalence Relation



A = Functions from $\mathbb{N}$ to $\mathbb{R}^+$

Define: $(f, g) \in R$ iff $f(n) = \Theta(g(n))$.

Claim R is an equivalence relation



① **Reflexive**

$\forall$ functions f, pick $c_1 = c_2 = 1$, $N = 1$

$$\forall n \geq N, \quad c_1 \cdot f(n) \leq f(n) \leq c_2 \cdot f(n)$$

Hence, $f(n) = \Theta(f(n)) \Rightarrow (f, f) \in R$

② Symmetric

Suppose $(f, g) \in R$ (want to prove $(g, f) \in R$)

$$\left.\begin{aligned} & f(n) = \Theta(g(n)) \\ & \exists c_1, c_2 > 0,\ N > 0 \\ & \forall n \geq N, \quad c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \end{aligned}\right\} \text{ by definition}$$

$$\underbrace{\frac{1}{c_2}}_{=\hat{c}_1} \cdot f(n) \leq g(n) \leq \underbrace{\frac{1}{c_1}}_{=\hat{c}_2} \cdot f(n)$$

$\Rightarrow g(n) = \Theta(f(n))$ (choice $\hat{c}_1 = \frac{1}{c_2}$, $\hat{c}_2 = \frac{1}{c_1}$, same N)

③ Transitive.

Suppose $(f,g) \in R$ and $(g,h) \in R$ (want to prove $(f,h) \in R$)

$f(n) = \Theta(g(n))$

$\exists\, c_1, c_2 > 0,\ \exists N,\quad \forall n \geq N,\quad c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$

$g(n) = \Theta(h(n))$

$\exists\, d_1, d_2 > 0,\ \exists M,\ \forall n \geq M,\ d_1 \cdot h(n) \leq g(n) \leq d_2 \cdot h(n)$

$$\frac{1}{c_2 \cdot d_2} f(n) \leq \frac{1}{d_2} \cdot g(n) \leq h(n) \leq \frac{1}{d_1} \cdot g(n) \leq \frac{1}{c_1 \cdot d_1} \cdot f(n)$$

Pick $\hat{c}_1 = \dfrac{1}{c_2 \cdot d_2}$, $\hat{c}_2 = \dfrac{1}{c_1 d_1}$

Pick $\hat{N} = \max(N, M)$

$\Rightarrow \forall n \geq \hat{N}. \quad \hat{c}_1 \cdot f(n) \leq h(n) \leq \hat{c}_2 \cdot f(n).$

$$h(n) = \Theta(f(n)) \Rightarrow f(n) = \Theta(h(n))$$

$(h, f) \in R$ $\qquad\qquad$ $(f, h) \in R.$

already proved symmetric.