COMP2121: Discrete Mathematics
Homework 1 **Due Date:** Oct 5, 2018

Rules: Discussion of the problems is permitted, but writing the assignment together is not (i.e. you are not allowed to see the actual pages of another student).

Course Outcomes

- **[O1. Abstract Concepts]**
- **[O2. Proof Techniques]**
- **[O3. Basic Analysis Techniques]**

1. (10 pt) [O1] Express the following statements using the propositions p "He has the ID card", q "He has the password", and r "He opens the door" together with logical connectives.

   (a) "He has the ID card and opens the door."

   (b) "He does not have the password and opens the door."

   (c) "If he has the password and does not have the ID card, he does not open the door."

   (d) "He opens the door if and only if he has the ID card and has the password."

   (e) "If he does not open the door, then he either does not have the ID card or does not have the password."

2. (10 pt) [O1, O2] Show that $(p \to q) \to r$ and $\neg r \to (p \wedge \neg q)$ are logically equivalent.

   (a) using truth tables.

   (b) using logical equivalence.

3. (15 pt) **[O1]** Consider the following conjecture in number theory:

   Every even number is the difference of two primes.

   (a) Express the statement in terms of quantifiers, variable(s), equality and inequality symbols ($<, >, =$), logical operators ($\wedge, \vee, \to$) and predicates $P(n)$: n is a prime number and $E(n)$: n is an even number.

   (b) Express the negation of (a) **without** using the logical operator $\neg$.

   [Be careful to define the domain(s) of your variable(s)]

4. (10 pt) **[O2]** Assuming the truth of the theorem which states that "$\sqrt{n}$ is irrational whenever n is a positive integer that is not a perfect square," prove that $\sqrt{3} + \sqrt{5}$ is irrational. (Hint: Suppose a is irrational and b is an integer, you can prove that $a + b$ is irrational by contradiction.)

5. (18 pt) **[O2, O3]** Use mathematical induction to prove the following.

(a) Prove that $H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$ where $H_n = 1 + 1/2 + \cdots + 1/n$ denotes the n-th harmonic number.

(b) Prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer.

6. (10 pt) **[O1, O2]** Decide whether the following statements about big-O notation are true or not.

   (a) Let $f(n) = \sqrt{n} + 5$, then $f(n) = \Omega(\log n)$.

   (b) Let $f(n) = n \log n - 4$, then $f(n) = O(n^2)$.

   (c) Let $f(n) = n + \log n$, then $f(n) = O(\log^2 n)$.

   (d) Let $f(n) = 2n^3 + 5n^2 \log n + 4$, then $f(n) = O(n^3)$.

   (e) Let $f(n) = 5 \log n + \sqrt{n} + 2$, then $f(n) = \Theta(\sqrt{n})$.

7. (15 pt) **[O2, O3]** A relation R is defined on the set Z of integers by xRy if $3x - 7y$ is even. Prove that R is an equivalence relation.

8. (12 pt) **[O2, O3]** Let $f : X \to Y$ be a function. Show that the following statements are equivalent.

   (a) There exists a function $g : Y \to X$ such that $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$.

   (b) f is a bijection.

   (Hint: Prove (a)$\Rightarrow$(b) and (a)$\Leftarrow$(b). When proving (a)$\Leftarrow$(b), if you use the inverse of function f, you should prove that it exists first.)