COMP2121: Discrete Mathematics
Homework 3 **Due Date:** 7pm, 30 Nov, 2018

Rules: Discussion of the problems is permitted, but writing the assignment together is not (i.e. you are not allowed to see the actual pages of another student).

Course Outcomes

- [**O1. Abstract Concepts**]
- [**O2. Proof Techniques**]
- [**O3. Basic Analysis Techniques**]

1. (6 points) [**O1**] How many vertices and how many edges does each of these graphs have?
   (a) Complete graph K_n.
   (b) Cycle graph C_n.
   (c) Complete bipartite graph $K_{n,m}$.
2. (9 points) [**O1**] For which values of $n \geq 3$ do these graphs have an Euler circuit?
   (a) Complete graph K_n.
   (b) Cycle graph C_n.
   (c) Complete bipartite graph $K_{n,n}$.
3. (10 points) [**O2**] Show that a simple graph G with n vertices is connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.
4. (14 points) [**O2**] Prove the following statements:
   (a) Complete graph K_n is not a planar graph when $n \geq 5$.
   (b) Complete bipartite graph $K_{n,n}$ is not a planar graph when $n \geq 3$.
5. (36 points) [**O3**] Suppose K_n is a complete graph whose vertices are indexed by $[n] = \{1, 2, 3, ..., n\}$, where $n \geq 4$. In this question, a cycle is identified solely by the collection of edges it contains; there is no particular orientation or starting point associated with a cycle. (Give your answers in terms of n for the following questions.)
   (a) How many Hamiltonian cycles are there in K_n?
   (b) How many Hamiltonian cycles in K_n contain the edge $\{1, 2\}$?
   (c) How many Hamiltonian cycles in K_n contain both the edges $\{1, 2\}$ and $\{2, 3\}$?
   (d) How many Hamiltonian cycles in K_n contain both the edges $\{1, 2\}$ and $\{3, 4\}$?
   (e) Suppose that M is a set of $k \leq \frac{n}{2}$ edges in K_n with the property that no two edges in M share a vertex. How many Hamiltonian cycles in K_n contain all the edges in M? Give your answer in terms of n and k.

(f) How many Hamiltonian cycles in K_n **do not** contain any edge from $\{1, 2\}$, $\{2, 3\}$ and $\{3, 4\}$?

6. (10 points) **[O1, O2]** Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuit of length 4 or less. Prove that if $v \geq 4$ then $e \leq \frac{5}{3}v - \frac{10}{3}$.

7. (15 points) **[O1, O3]** Suppose there is a connected planar simple graph G with v vertices such that all its regions are triangles (a cycle consisting of three edges).

   (a) (3 points) Into how many regions does a representation of the planar graph G split the plane?

   (b) (12 points) Suppose the vertices of the planar graph G are colored in three colors. A region is called to be tricolored (or bicolored) if its vertices are colored in exactly three (or two) different colors. Similarly, a monocolored region is the one with all its vertices colored in exactly one color. Prove that the number of tricolored regions is always even no matter how the vertices are colored.

   (Hint: If you place a new vertex inside a region (triangle) of G and connect it with all vertices of that region, then all regions are still triangles and the parity of the total number of regions stays the same.)