COMP2121A: Discrete Mathematics
Problem Solving Session 2 Sep 18, 2017

Review. Proof methods, $p \to q \Leftrightarrow \neg p \vee q$

1. direct proof
2. proof by contradiction, assume $p \wedge \neg q$, try to find a contradiction
3. proof by contrapositive, prove $\neg q \to \neg p$
4. mathematical induction, show that $P(x)$ is true for all $x \in \bigcup_{i \geq 0} S_i$
   (a) base case, prove $P(x)$ is true for all $x \in S_0$
   (b) inductive step, given $P(x)$ is true for all $x \in \bigcup_{0 \leq i \leq k-1} S_i$, prove $P(x)$ is true for all $x \in S_k$

Questions.

1. Given a real number x and an positive integer n, show an efficient method to evaluate x^n with only multiplications and additions.

2. Define $f(n) = 1^3 + 2^3 + 3^3 + \ldots + n^3$. Use mathematical induction to prove that $f(n) = \left[\frac{n\cdot(n+1)}{2}\right]^2$ for all positive integers.

3. Prove the following statement. There exist irrational numbers x and y such that x^y is rational. (Hint: Consider $\sqrt{2}^{\sqrt{2}}$. Is it rational or not?)

4. Given a finite set A of n points on the plane (2-dimensional space) such that for any two points x, y in A, the line containing x and y must contain another point z in A. Prove that all points in A are on the same line.[1]

   (a) Is the following proof (induction on the number of points) correct? If not, where is the bug?

   - Base case: for point set of size 3 the statement is true.
   - Inductive step: assume this statement is true for point set of size $k \geq 3$. Consider the case when we have a point set A of size $k+1$. We argue as follows.

     i. Pick A' of k points from the given point set A. Let x be the other point in A but not in A'.

     ii. By *induction hypothesis*, points in A' are on the same line.

     iii. Pick any y in A', the line going through x, y contains another point z in A.

     iv. Thus, x, y and z are on the same line.

     v. So x and all points in A' are on the same line.

   (b) Can you give a proof by contradiction?

[1] This is equivalent to Sylvester-Gallai theorem, which is named after James Joseph Sylvester, who posed it as a problem in 1893, and Tibor Gallai, who published one of the first proofs of this theorem in 1944.