COMP2121A: Discrete Mathematics
Problem Solving Session 26/30 Oct, 2018

1. **Balls and Bins** Suppose each of n balls are thrown independently into n bins uniformly at random. Let X be the number of empty bins. Compute $E[X]$.

2. Suppose $n \geq 2$ and m are positive integers. There are m identical coins to be distributed among n persons.

   We describe a procedure called UNFAIR to divide the coins. The first person comes along and an integer x is picked uniformly at random from $\{0, 1, 2, \ldots, m\}$. Then, the first person takes x coins and goes home. In general, when the ith person comes along (where $i < n$), and there are r coins left, an integer y is picked uniformly at random from $\{0, 1, 2, \ldots, r\}$ and the ith person goes home with y coins. The nth (last) person just takes whatever that is left.

   Define X_i to be the number of coins the ith person takes.

   (a) Compute $E[X_1]$, the expected number of coins the first person receives.

   (b) Suppose $n \geq 3$. Given that the first person receives x coins, what is the expected number of coins the second person receives? (Compute $E[X_2|X_1 = x]$.)

   (c) Assume $n \geq 3$. Compute $E[X_2]$.

   (d) For general $i \leq n$, compute $E[X_i]$.

We next consider another procedure called FAIR. First, compute the set S of all integer solutions to the equation $x_1 + x_2 + x_3 + \ldots + x_n = m$, where each $x_i \geq 0$. A solution $(x_1, x_2, \ldots, x_n)$ is picked uniformly at random from S, and for each $1 \leq i \leq n$, the ith person receives x_i coins.

(a) What is the size of S?

(b) Suppose X_1 is the number of coins received by the first person. What is the probability that $X_1 = k$, where $0 \leq k \leq m$? Express your answer in terms of n, m and k.

(c) Prove that for all positive integers $n \geq 2$ and $m \geq 1$,

$$\sum_{k=0}^{m} k \cdot \binom{m+n-k-2}{n-2} = \frac{m}{n} \cdot \binom{m+n-1}{m}.$$

3. Let $B = \mathsf{Bin}(n, p)$, i.e., flipping n biased coins, each having heads with probability p. Compute $E[B^2]$.

   (For general $k \geq 2$, how to compute $E[B^k]$?)