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Enhancing Levin's method for computing quadric-surface intersections

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Abstract

Levin's method produces a parameterization of the intersection curve of two quadrics in the form

$$\mathbf{p}(u) = \mathbf{a}(u) \pm \mathbf{d}(u)\sqrt{s(u)},$$

where $\mathbf{a}(u)$ and $\mathbf{d}(u)$ are vector valued polynomials, and $s(u)$ is a quartic polynomial. This method, however, is incapable of classifying the morphology of the intersection curve, in terms of reducibility, singularity, and the number of connected components, which is critical structural information required by solid modeling applications. We study the theoretical foundation of Levin's method, as well as the parameterization $\mathbf{p}(u)$ it produces. The following contributions are presented in this paper: (1) It is shown how the roots of $s(u)$ can be used to classify the morphology of an irreducible intersection curve of two quadric surfaces. (2) An enhanced version of Levin's method is proposed that, besides classifying the morphology of the intersection curve of two quadrics, produces a rational parameterization of the curve if the curve is singular. (3) A simple geometric proof is given for the existence of a real ruled quadric in any quadric pencil, which is the key result on which Levin's method is based. These results enhance the capability of Levin's method in processing the intersection curve of two general quadrics within its own self-contained framework.

Keywords: Quadric surface; Intersection; Stereographic projection

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1. Introduction

Quadric surfaces are the simplest curved surfaces and are widely used in computer graphics and solid modeling. Computing quadric surface intersection curves (QSIC) is an important operation in computing the boundary representation of a solid. The goal of this paper is to enhance Levin's method for computing a QSIC so as to make it capable of computing the structural information of the QSIC. Specifically, we show how the roots of a discriminant polynomial $s(u)$ computed by Levin's method can be used to classify the morphology of an irreducible intersection curve of two quadric surfaces, in terms of its reducibility, singularity, and the number of connected components. Based on this result, we present an enhanced version of Levin's method that is capable of classifying the morphology of a QSIC and producing a rational parameterization if the QSIC is singular. Furthermore, we give a concise geometric proof for the existence of a real ruled quadric in any real quadric pencil, which is the key result on which Levin's method is based.

The remainder of the paper is organized as follows. In the rest of this section we discuss the basic properties of the QSIC and review Levin's method as well as other related work. In Section 2, procedures are described for detecting and processing a reducible QSIC, focusing on a remedy to make Levin's method capable of detecting and parameterizing properly a reducible but nonplanar QSIC, comprising a line and a space cubic curve. In Section 3, with the aid of a stereographic projection, we obtain characterizations of different morphologies of irreducible QSIC's in terms of the roots of a discriminant polynomial computed by Levin's method. In Section 4, based on the preceding analysis, we present an enhanced version of Levin's method for classifying and parameterizing a general QSIC. The paper concludes in Section 5 with a summary of our work. In Appendix A we give a concise geometric proof for the existence of a real ruled quadric in any real quadric pencil.

1.1. Morphologies of QSIC's

Let CP^3, RP^3, and RA^3 denote, respectively, 3D complex projective space, 3D real projective space, and 3D real affine space. Every quadric discussed in this paper is assumed to be defined by the zero set of a quadratic form $X^{\mathrm{T}}AX$ in CP^3, where X is a 4D column vector consisting of the homogeneous coordinates of a point and A is a 4×4 real symmetric matrix. The intersection curve of two quadrics $X^{\mathrm{T}}AX = 0$ and $X^{\mathrm{T}}BX = 0$ comprises all the points in CP^3 that satisfy both of these equations.

The full classification of the morphology of a QSIC in CP^3 can be found in classical texts on algebraic geometry and solid geometry (Baker, 1923; Sommerville, 1947; Semple and Kneebone, 1952). The intersection curve of two quadrics (QSIC) is a space quartic curve of the first species. A QSIC is *reducible* if it contains some linear, quadratic (conic), or cubic components, whose degrees sum to 4; otherwise it is called *irreducible*. The linear or conic components of a reducible QSIC can be real or imaginary, but in the case where the QSIC consists of a line and a space cubic, the line and the space cubic are real. A QSIC is planar if its components lie on one or two planes. All planar QSIC's are reducible. Nonplanar QSIC's include all irreducible QSIC's and those reducible QSIC's that comprise a line and a space cubic.

A QSIC is called *singular* if it has a singular point, i.e., a point at which the tangent line is not uniquely defined; otherwise it is *nonsingular*. A reducible QSIC is always singular, but an irreducible QSIC can be singular or nonsingular. A singular but irreducible QSIC has exactly one double singular point of three possible types, i.e., acnode, crunode, or cusp, and is a rational curve of degree 4. Such a QSIC may have only one real acnode without any other real regular point. A nonsingular QSIC can have zero, one, or two connected components in RP^3. A nonsingular QSIC does not permit a rational parameterization, but can be parameterized with a square root or an elliptic function (Farouki et al., 1989). The following diagram shows a hierarchy of all QSIC's with respect to their reducibility, planarity, and singularity:

$$\text{QSIC}\begin{cases}\text{reducible QSIC}\begin{cases}\text{planar QSIC: comprising lines or conics;}\\ \text{reducible but nonplanar QSIC: a real line and a real space cubic;}\end{cases}\\ \text{irreducible QSIC}\begin{cases}\text{singular QSIC with a real acnode, crunode, or cusp;}\\ \text{nonsingular QSIC with zero, one, or two components in } RP^3.\end{cases}\end{cases}$$

1.2. Related work

The Segre characteristic provides a useful characterization of different morphologies of a QSIC in CP^3 (Bromwich, 1906; Farouki et al., 1989). However, such classical results are often not sufficient because CAD and graphics applications often require the classification and representation of the QSIC to take place in real (projective, affine, or Euclidean) space. For instance, the single Segre characteristic [(11)11] is assigned to a QSIC comprising two conics touching at two distinct points, but there are four different morphologies in RP^3 in this case, depending on whether the two conics are real or imaginary and whether the two common points are real or imaginary. Similarly, nonsingular QSIC's with different numbers of connected components in RP^3 all correspond to the same Segre characteristic [1111]. Some recent results are reported in (Tu et al., 2002) that use the roots of the characteristic equation $|\lambda A - B| = 0$ to distinguish different types of nonsingular QSIC's.

Computational requirements for QSIC's vary from tracing the curve for graphics display to deriving geometric and topological information for geometric processing. A variety of methods can be found in the literature for computing QSIC's. These methods provide different levels of information or have different assumptions on the kinds of quadrics or QSIC's that can be handled. There are notably two different approaches to computing the QSIC: the geometric approach and the algebraic approach. Methods using the geometric approach normally exploit special geometric properties of a special class of quadrics to yield robust procedures for computing the QSIC (Miller, 1987; Piegl, 1989; Shene and Johnstone, 1994; Miller and Goldman, 1995). Such a special class usually includes natural quadrics, i.e., spheres, circular cones and cylinders, plus pairs of planes. The focus on natural quadrics is justified by their frequent occurrence in engineering applications. In the following we review only methods using the algebraic approach, since these methods normally accept arbitrary quadrics.

Levin's method is one of the early methods for computing the general QSIC of two arbitrary quadrics. It produces a parameterization of the QSIC with a square-root function but does not yield information about the reducibility or singularity of the QSIC. Hence, this method is mainly a technique for tracing the intersection curve. Since the goal of the present paper is to improve Levin's method, we will review Levin's method in more detail in Section 1.3.

Levin's method is implemented by Sarraga in GMSOLID for computing the intersection curves of natural quadrics (Sarraga, 1983). Sarraga also attempts to give a geometric interpretation of the zeros of a quartic discriminant polynomial $s(u)$ generated in Levin's method for segmenting a QSIC but does not provide a complete analysis regarding the reducibility, the singularity, and the number of connected components of a QSIC. Ocken, Schwartz, and Sharir (1987) propose to use a projective transformation to reduce two input quadrics to simple canonical forms whose intersection curve can easily be found. It is not clear whether a complete classification of a QSIC can be accomplished by their results, since their procedures are not thoroughly analyzed and some cases are missing; for instance, the case of two quadrics intersecting in a line and a space cubic is not accounted for.

Degenerate QSIC's of arbitrary quadrics are studied in (Farouki et al., 1989). A degenerate QSIC is detected by the vanishing of the discriminant of the equation $|\lambda A + B| = 0$ of two quadrics $X^{\mathrm{T}}AX = 0$ and $X^{\mathrm{T}}BX = 0$, and is further classified by projecting the QSIC to a planar quartic curve and analyzing this quartic curve. Wilf and Manor use the Segre characteristic to assist with classifying a QSIC before invoking Levin's method to yield a proper parameterization of the QSIC (Wilf and Manor, 1993). Using the Segre characteristic allows them to properly generate all linear components of a QSIC that might be missed by Levin's method and to generate a rational parameterization for a singular QSIC. However, this method still cannot count the number of connected components of a nonsingular QSIC.

More recently, Wang, Joe, and Goldman (2002) compute the QSIC by first obtaining and analyzing a planar cubic curve that is the image of the QSIC under a general stereographic projection. This method accepts arbitrary input quadrics, classifies the morphology of a general QSIC, and yields a rational parameterization for a singular QSIC; however, an initial point on the QSIC needs to be computed first in order to invoke the method.

Other methods for computing the intersection curves of general parametric or algebraic surfaces can also be applied to computing the QSIC; see, for example, (Abhyankar and Bajaj, 1989; Garrity and Warren, 1989). However, since these methods are typically devised for more general classes of surfaces, they normally do not take into account the specific algebraic properties of quadric surfaces. Therefore they provide an algebraic representation of a QSIC that is usually more complicated than the one given by Levin's method.

1.3. Levin's method

Levin's method is a procedure for parameterizing an arbitrary QSIC with a square root in the form

$$\mathbf{p}(u) = \mathbf{a}(u) \pm \mathbf{d}(u)\sqrt{s(u)},$$

where $\mathbf{a}(u)$ and $\mathbf{d}(u)$ are vector valued polynomials and $s(u)$ is a quartic polynomial. Levin published two papers on this method (Levin 1976, 1978). The first paper presents the basic idea of using a real ruled quadric, called the *parameterization surface*, in the pencil of two quadrics to find the QSIC and proves the existence of such a parameterization surface. The second paper discusses implementation issues and describes the procedure in detail. Levin's method has inspired several subsequent papers on its application and improvement (Sarraga, 1983; Wilf and Manor, 1993), as well as the present paper.

Two distinct quadrics $\mathcal{A}$: $X^{\mathrm{T}}AX = 0$ and $\mathcal{B}$: $X^{\mathrm{T}}BX = 0$, i.e., their coefficient matrices are linearly independent, span a pencil of quadrics: $X^{\mathrm{T}}(\lambda A + B)X = 0$. The intersection curve of $\mathcal{A}$ and $\mathcal{B}$ is called the *base curve* of the pencil. It is evident that any member of the pencil passes through the base curve and any two distinct members of the pencil intersect at the base curve. Hence, the intersection curve of $\mathcal{A}$ and $\mathcal{B}$ can be computed by intersecting any two distinct quadrics in the pencil $X^{\mathrm{T}}(\lambda A + B)X = 0$.

Levin's method is based on the critical observation that there exists a real ruled quadric $\mathcal{S}$: $X^{\mathrm{T}}SX = 0$ in the pencil $X^{\mathrm{T}}(\lambda A + B)X = 0$. This surface is called a *parameterization surface*. The quadric $\mathcal{S}$ can be a pair of planes, a singly ruled quadric (i.e., a cone or a cylinder), or a doubly ruled quadric (i.e., a one-sheet hyperboloid or a hyperbolic paraboloid). There is only one family of straight lines on a singly ruled quadric $\mathcal{S}$, all passing though the vertex of $\mathcal{S}$; the vertex is a point at infinity if $\mathcal{S}$ is a cylinder. There are two reguli on a doubly ruled quadric $\mathcal{S}$ (Semple and Kneebone, 1952;

Pottmann and Wallner, 2001); two distinct lines from the same regulus are skew and two lines from different reguli intersect.

Recall that linear combinations of the homogeneous coordinates of two distinct points X_0 and X_1 generate all points on the line connecting X_0 and X_1. So a ruled quadric $\mathcal{S}$ can be parameterized by

$$\mathbf{q}(u, v) = \mathbf{b}(u) + v\mathbf{d}(u), \tag{1}$$

where $\mathbf{q}(u, v)$, $\mathbf{b}(u)$, and $\mathbf{d}(u)$ are vector valued polynomials in homogeneous coordinates. If $\mathcal{S}$ is singly ruled, $\mathbf{b}(u)$ is the vertex of $\mathcal{S}$ and $\mathbf{d}(u)$ is a proper conic on $\mathcal{S}$, i.e., $\mathbf{b}(u)$ has degree 0 and $\mathbf{d}(u)$ has degree 2. (Note that when $\mathcal{S}$ is a cylinder, $\mathbf{b}(u)$ is not a direction, but a 4D vector for the homogeneous coordinates of the singular point of $\mathcal{S}$ at infinity.) If $\mathcal{S}$ is doubly ruled, $\mathbf{b}(u)$ and $\mathbf{d}(u)$ are two skew lines from the same regulus, so both are of degree 1. In general, $\mathbf{b}(u)$ and $\mathbf{d}(u)$ are called the generating curves for parameterizing the ruled surface $\mathcal{S}$ through (1).

Since $\mathcal{A}$ or $\mathcal{B}$ are distinct surfaces, we can assume, without loss of generality, that the ruled surface $\mathcal{S}$ is distinct from $\mathcal{A}$. To find the base curve of the pencil $X^{\mathrm{T}}(\lambda A + B)X = 0$, we substitute $\mathbf{q}(u, v)$ for X in $X^{\mathrm{T}}AX = 0$ and obtain

$$c_2(u)v^2 + 2c_1(u)v + c_0(u) = 0, \tag{2}$$

where

$$c_2(u) = \mathbf{d}(u)^{\mathrm{T}}A\mathbf{d}(u), \qquad c_1(u) = \mathbf{b}(u)^{\mathrm{T}}A\mathbf{d}(u), \qquad c_0(u) = \mathbf{b}(u)^{\mathrm{T}}A\mathbf{b}(u).$$

This equation has the solution

$$v = \frac{-c_1(u) \pm \sqrt{s(u)}}{c_2(u)},$$

where

$$s(u) = c_1^2(u) - c_2(u)c_0(u) \tag{3}$$

is the discriminant of (2) and a quartic polynomial in u. Substituting the above solution for v in $\mathbf{q}(u, v)$ yields the following homogeneous parameterization of the QSIC

$$\mathbf{p}(u) = c_2(u)\mathbf{b}(u) + \left[-c_1(u) \pm \sqrt{s(u)}\,\right]\mathbf{d}(u) = \mathbf{a}(u) \pm \sqrt{s(u)}\,\mathbf{d}(u), \tag{4}$$

where

$$\mathbf{a}(u) = c_2(u)\mathbf{b}(u) - c_1(u)\mathbf{d}(u).$$

Levin's method, as originally proposed, is mainly a technique for tracing a QSIC and is incapable of providing the structural information of the QSIC, such as reducibility, singularity, and the number of connected components. See also (Farouki et al., 1989; Wilf and Manor, 1993) for their remarks about the difficulties with Levin's method. Specifically, Levin's method has the following problems:

(1) The reducibility of a QSIC is not detected properly. For example, as pointed out in (Wilf and Manor, 1993), if a QSIC consists of a line ℓ_1 and a space cubic, the linear component ℓ_1 may be missed by Levin's method. We provide a detailed analysis of, as well as a remedy for, this problem in Section 2.2.
(2) The singularity of a QSIC is not detected and classified. As a consequence, Levin's method may fail to yield a rational parameterization for a singular QSIC, which is a rational curve.

(3) The number of connected components of a nonsingular QSIC in RP^3 is not computed. The enumeration and identification of different connected components of a QSIC is useful topological information; for instance, when combined with the points at infinity on the QSIC, this information can be used for segmenting the QSIC in AR^3, the 3D real affine space, since only finite segments of a curve are used in practice (Sarraga, 1983). Note that none of the existing methods for processing the QSIC has addressed the problem of counting or identifying the number of connected components of a nonsingular QSIC.

In addition, the algebraic proof by Levin for the existence of a real ruled surface in any quadric pencil is lengthy and involved (Levin, 1976); it takes four lemmas and fills nearly two double-columned pages. We give a much shorter and more accessible geometric proof of this fact in Appendix A.

2. Reducible intersection curves

2.1. Planar QSIC's

There exist well-studied solutions to the problem of computing planar QSIC's (Levin, 1978; Sarraga, 1983; Miller and Goldman, 1995); the problem is basically reduced to computing the intersection between a quadric and two planes. Hence, we turn our attention to the case of nonplanar reducible QSIC's for which, in general, Levin's method is known to fail.

2.2. Nonplanar and reducible QSIC's

A reducible but nonplanar QSIC consists a real space cubic curve and a real line. The line may intersect the space cubic in two distinct real or complex conjugate points, or may be tangent to the space cubic. It is first pointed out in (Wilf and Manor, 1993) that Levin's method cannot, in general, produce a proper parameterization of the QSIC in this case, and that the linear component may be missing from the parameterization entirely.

Consider a QSIC consisting of a line ℓ_1 and a space cubic. Suppose that $\mathcal{S}$ is doubly ruled. Let the generating curve $\mathbf{b}(u)$ for parameterizing $\mathcal{S}$ by $\mathbf{q}(u, v)$ (1) be a line r_0 in the regulus $\mathcal{R}$ on $\mathcal{S}$. Suppose further that the linear component ℓ_1 lies in the other regulus $\mathcal{L}$ on $\mathcal{S}$. Then ℓ_1 is given by $\mathbf{q}(u_0, v)$ for some value u_0. Thus Eq. (2) is satisfied by all v, with $u = u_0$. It follows that the three coefficients $c_i(u)$ of Eq. (2) vanish simultaneously at u_0. This means that, with the parameterization $\mathbf{p}(u)$ of the QSIC given by (4), the single point $\mathbf{p}(u_0)$ corresponds to the entire line ℓ_1; that is, the line ℓ_1 is not represented properly by the parameterization $\mathbf{p}(u)$.

A similar analysis shows that the same problem also arises when $\mathcal{S}$ is singly ruled. Hence, instead of attempting to analyze the parameterization $\mathbf{p}(u)$ in this case, a QSIC consisting of a line and space cubic should be detected before $\mathbf{p}(u)$ is computed. And, if such a QSIC is detected, one should first extract the linear component and then proceed to parameterize the residual cubic space curve.

A QSIC consisting of a space cubic and a line can be detected as follows. Suppose first that the parameterization surface $\mathcal{S}$ is singly ruled. Form the quadratic equation (2). Then the QSIC has a linear component if and only if the three coefficients $c_i(u)$ of Eq. (2) vanish simultaneously for some u_0. Since $c_0(u) = \mathbf{b}(u)^{\mathrm{T}} A \mathbf{b}(u)$ is a constant, one may check if it is zero. If it is nonzero, then the QSIC does not have a linear component; if it is zero, then the QSIC has a linear component if and only if the other two coefficients $c_1(u)$ and $c_2(u)$ have a common nonconstant factor. Note that $c_1(u)$ and $c_2(u)$ are polynomials of degree 2 and degree 4, respectively. They have a common nonconstant factor if and only if $\mathrm{Res}(c_1(u), c_2(u)) = 0$, or equivalently, if and only if $\mathrm{GCD}(c_1(u), c_2(u))$ is not constant.

When $\mathcal{S}$ is doubly ruled, one may first derive two parameterizations of $\mathcal{S}$, with the generating line $\mathbf{b}(u)$ chosen from each of the two reguli on $\mathcal{S}$. Then the QSIC contains a linear component if and only if the three coefficients $c_0(u)$, $c_1(u)$, and $c_2(u)$ of Eq. (2) have a common nonconstant factor for either of these two parameterizations of $\mathcal{S}$. Note that in this case the $c_i(u)$ are quadratic polynomials. It is easy to show that the $c_i(u)$, $i = 0, 1, 2$, have a common factor if and only if the three resultants $\mathrm{Res}(c_0(u), c_2(u))$, $\mathrm{Res}(c_1(u), c_2(u))$, and $\mathrm{Res}(c_0(u) + c_1(u), c_2(u))$ are zero. Alternatively, one may compute $\mathrm{GCD}(\mathrm{GCD}(c_0(u), c_1(u)), c_2(u))$ to detect if the $c_i(u)$, $i = 0, 1, 2$, have a common nonconstant factor.

3. Irreducible intersection curves

3.1. Stereographic projection

The main idea in the analysis of an irreducible QSIC is to use a stereographic projection to project the QSIC to a planar cubic curve that has the same structure as the QSIC. To facilitate the construction of the stereographic projection, we first give two conditions that characterize a QSIC devoid of real regular points. Note that an irreducible QSIC cannot have more than one singular point.

The following two results are obvious.

Theorem 1. *If a QSIC is devoid of real points then $s(u) < 0$ for all u.*

Theorem 2. *If an irreducible QSIC has one real singular point but no other real points then $s(u) < 0$ for all u except for one value u_0 at which $s(u_0) = 0$.*

Since the case of an irreducible QSIC with no real regular points can be characterized by Theorems 1 and 2, in the following we will consider only QSIC's with real regular points. Let $\mathcal{S}$ denote a parameterization surface used in Levin's method, which is a ruled quadric in the pencil of two input quadrics $\mathcal{A}$ and $\mathcal{B}$. We define a stereographic projection $\mathbf{M}$ from the ruled quadric $\mathcal{S}$ that maps the QSIC to a planar cubic curve $\mathcal{H}$ on a plane $\mathcal{P}$ in RP^3. (See (Sommerville, 1947; Wang et al., 1997) for a stereographic projection on a general quadric surface and its properties.) We will see that this projection preserves a number of algebraic and topological properties of the QSIC, i.e., the QSIC and $\mathcal{H}$ have the same reducibility, the same type of singularity, and the same number of connected components. In addition, the projection $\mathbf{M}$ maps a regulus on $\mathcal{S}$ to a pencil of lines centered at a point on $\mathcal{H}$. These properties of $\mathbf{M}$ are critical in performing a rigorous analysis of the geometric meaning of the polynomial $s(u)$, since the problem is translated to studying the intersection of a planar cubic curve with a pencil of lines.

First suppose that $\mathcal{S}$ is a doubly ruled quadric. See Fig. 1. Let $\mathcal{R}$ and $\mathcal{L}$ denote the two reguli on $\mathcal{S}$. Let N_0 be a real regular point on the QSIC. Let $r_0 \in \mathcal{R}$ and $l_0 \in \mathcal{L}$ be the two lines passing through the point N_0, and let r_0 be the line $\mathbf{q}(u, 0) = \mathbf{b}(u)$ used in the parameterization (1) of $\mathcal{S}$. Take N_0 as the center of a stereographic projection $\mathbf{M}$ from $\mathcal{S}$ to a plane $\mathcal{P}$ in RP^3 not passing through N_0. Suppose that the line

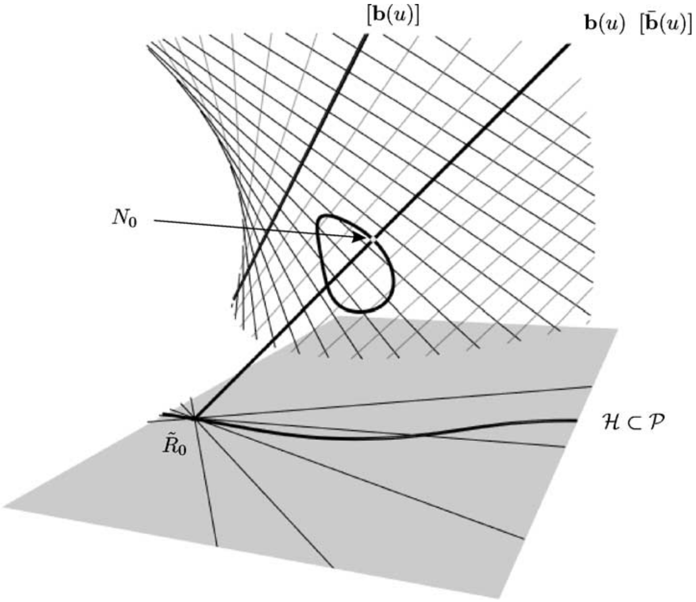

Fig. 1. A stereographic projection on $\mathcal{S}$.

r_0 is projected by **M** to a point $\tilde{R}_0$ in the plane $\mathcal{P}$. Then, since the center N_0 of **M** is a regular point of the QSIC, the QSIC is mapped by **M** to a planar cubic curve $\mathcal{H}$ passing through $\tilde{R}_0$ on $\mathcal{P}$ (Baker, 1923; Sommerville, 1947). That $\mathcal{H}$ is a cubic curve can easily be seen as follows. Let $\mathcal{P}'$ be a generic plane passing through the point N_0; thus the intersection between $\mathcal{P}'$ and $\mathcal{P}$ is a generic line, denoted by ℓ', on $\mathcal{P}$. Since the QSIC is a space quartic curve, $\mathcal{P}'$ intersects the QSIC at N_0 and three other free points. Clearly, the three free intersection points are in one-to-one correspondence with intersections between the line ℓ' and the curve $\mathcal{H}$. Hence, $\mathcal{H}$ is a planar cubic curve since it is intersected three times by a generic line ℓ' in the plane. Furthermore, all the lines in $\mathcal{L}$ are mapped to a pencil of lines $\tilde{\mathcal{L}}$ centered at the point $\tilde{R}_0 \in \mathcal{H}$ in the plane $\mathcal{P}$ (see Fig. 1).

When $\mathcal{S}$ is singly ruled, in a similar manner we may also define a stereographic projection **M** on $\mathcal{S}$. In this case, the base curve $\mathbf{d}(u)$ in (1) for parameterizing $\mathcal{S}$ is a conic, and $\mathbf{b}(u)$ is a constant vector for the vertex of $\mathcal{S}$ (see Section 1.3). Suppose that the conic $\mathbf{d}(u)$ passes through a real regular point N_0 on the QSIC. Take N_0 as the center of a stereographic projection **M** from $\mathcal{S}$ to a plane $\mathcal{P}$ in RP^3 not passing through N_0. Let the vertex $\mathbf{b}(u)$ of $\mathcal{S}$ be projected through N_0 to a point $\tilde{R}_0$ on $\mathcal{P}$. Then the QSIC is again projected by **M** to a planar cubic curve $\mathcal{H}$ passing through $\tilde{R}_0$ and all the lines on $\mathcal{S}$, denoted as a group by $\mathcal{L}$, are projected to a pencil of lines $\tilde{\mathcal{L}}$ centered at the point $\tilde{R}_0$ in the plane $\mathcal{P}$. Hence, we have defined a stereographic projection on the parameterization surface $\mathcal{S}$ when $\mathcal{S}$ is doubly ruled or singly ruled.

Assumption. (a) When the parameterization surface $\mathcal{S}$ is doubly ruled, the line $\mathbf{b}(u)$ in the parameterization of $\mathcal{S}$ given by (1) passes through the center N_0 of the stereographic projection **M**; or (b) when the parameterization surface $\mathcal{S}$ is singly ruled, the conic $\mathbf{d}(u)$ in the parameterization of $\mathcal{S}$ given by (1) passes through the center N_0 of the stereographic projection **M**.

Remark. The above assumption facilitates the subsequent discussion in this subsection. But this assumption will be dropped as a consequence of Lemma 3 in Section 3.2, since it is not essential to the validity of the final results listed in Theorem 11 in Section 3.4.

To recap, the projection **M** preserves the following algebraic and topological properties of the QSIC:

(1) The QSIC is nonsingular if and only if the cubic $\mathcal{H}$ is nonsingular;
(2) When they are nonsingular, the QSIC and the cubic $\mathcal{H}$ have the same number of connected components;
(3) When they are singular but irreducible, the QSIC and the cubic $\mathcal{H}$ both have one double point; furthermore, their double points have the same type, i.e., either an acnode, a crunode, or a cusp.

The parameterization (4) of the QSIC is mapped by **M** to the following parameterization of the planar cubic $\mathcal{H}$:

$$\tilde{\mathbf{p}}(u) \equiv \mathbf{M}\big(\mathbf{p}(u)\big) = c_2(u)\mathbf{M}\big(\mathbf{b}(u)\big) + \big[-c_1(u) \pm \sqrt{s(u)}\,\big]\mathbf{M}\big(\mathbf{d}(u)\big) = \tilde{\mathbf{a}}(u) \pm \sqrt{s(u)}\,\tilde{\mathbf{d}}(u), \tag{5}$$

where

$$\tilde{\mathbf{a}}(u) = c_2(u)\mathbf{M}\big(\mathbf{b}(u)\big) - c_1(u)\mathbf{M}\big(\mathbf{d}(u)\big) \quad \text{and} \quad \tilde{\mathbf{d}}(u) = \mathbf{M}\big(\mathbf{d}(u)\big).$$

When the parameterization surface $\mathcal{S}$ is doubly ruled, $\mathbf{M}(\mathbf{b}(u))$, which is the projection of the line $r_0 \in \mathcal{R}$, is the fixed point $\tilde{R}_0$, and $\mathbf{M}(\mathbf{d}(u))$, which is the projection of another line $r_1 \in \mathcal{R}$, is a line not passing through $\tilde{R}_0$ in the plane $\mathcal{P}$, since r_0 and r_1 are two skew lines in the same regulus $\mathcal{R}$; that is, $\mathbf{M}(\mathbf{b}(u))$ has degree 0 and $\mathbf{M}(\mathbf{d}(u))$ has degree 1. When $\mathcal{S}$ is singly ruled, $\mathbf{M}(\mathbf{b}(u))$ is also a fixed point, since it is the projection of the vertex of $\mathcal{S}$, and $\mathbf{M}(\mathbf{d}(u))$ is also a line, since it is the projection of the conic $\mathbf{d}(u)$ with the projection center N_0 located on $\mathbf{d}(u)$; hence, again, $\mathbf{M}(\mathbf{b}(u))$ has degree 0 and $\mathbf{M}(\mathbf{d}(u))$ has degree 1.

The above parameterization $\tilde{\mathbf{p}}(u)$ (5) of the cubic $\mathcal{H}$ is actually the same parameterization of $\mathcal{H}$ that can be obtained by intersecting $\mathcal{H}$ with the pencil of the lines $\tilde{\mathcal{L}}$: $\tilde{R}_0 + v\mathbf{M}(\mathbf{d}(u))$ centered at the point $\tilde{R}_0$. Let $\tilde{\ell} \in \tilde{\mathcal{L}}$ denote a line that is the image of a line $\ell \in \mathcal{L}$ under the projection **M**. Then the two intersection points I_0 and I_1 between a line in $\ell \in \mathcal{L}$ and the QSIC correspond under the projection **M** to the two intersection points $\tilde{I}_0$ and $\tilde{I}_1$ between $\tilde{\ell}$ and the cubic curve $\mathcal{H}$. Moreover, (i) I_0 and I_1 are two distinct real points if and only if $\tilde{I}_0$ and $\tilde{I}_1$ are two distinct real points; and (ii) I_0 and I_1 collapse into a double real point if and only if $\tilde{I}_0$ and $\tilde{I}_1$ collapse into a double real point. Further, from (5) we see that the sign of $s(u)$ determines whether the two intersection points $\tilde{I}_0$ and $\tilde{I}_1$ of a line $\tilde{\ell} \in \tilde{\mathcal{L}}$ and the cubic $\mathcal{H}$ are two distinct real points or two complex conjugate points. In particular, the vanishing of $s(u)$, as the discriminant of (2), signals that the points $\tilde{I}_0$ and $\tilde{I}_1$ form a double real point, a case that occurs when the line $\tilde{\ell}$ is tangent to the cubic $\mathcal{H}$ at a point other than $\tilde{R}_0$ or when the line $\tilde{\ell}$ passes through a double point of a singular planar cubic $\mathcal{H}$.

Since planar cubic curves are well understood (Salmon, 1934; Bix, 1998), the intersection configurations between a planar cubic and a pencil of lines centered on the cubic can be studied in an exhaustive and rigorous manner. This investigation can then lead to a thorough analysis of the geometric interpretation of the zeros of $s(u)$ in connection with the planar cubic H, and hence also in connection with the QSIC through the stereographic projection **M** that relates the cubic $\mathcal{H}$ to the QSIC.

Because the intersection properties we are concerned with here are not affected by projective transformations in the plane $\mathcal{P}$, we may simplify our discussion by considering only the *normal form* of an irreducible planar cubic under projective transformations. The normal form of an irreducible planar cubic curve under projective transformation is $y^2 = x^3 + px + q$ (Bix, 1998). There are five topologically different species of an irreducible planar cubic in the normal form. Figs. 2–7 show singular but irreducible cubics with three different types of double singular points. Figs. 8–10 show two kinds of nonsingular cubics with one and two connected components, respectively. In the next two subsections we consider the intersection between the pencil $\tilde{\mathcal{L}}$ and the planar cubic $\mathcal{H}$, assuming in turn that $\mathcal{H}$ is of each of these five types.

3.2. Singular and irreducible intersection curves

In this section we consider a singular and irreducible QSIC. Since the center of the projection $\mathbf{M}$ is chosen to be at a regular point of a QSIC, a singular QSIC is projected to a planar cubic curve with one double point, which is an acnode, a crunode, or a cusp.

Lemma 3. *Given two quadrics $\mathcal{A}$ and $\mathcal{B}$, suppose the parameterization surface $\mathcal{S}$ in Levin's method has two parameterizations*

$$\mathbf{q}(u, v) = \mathbf{b}(u) + v\mathbf{d}(u)$$

and

$$\bar{\mathbf{q}}(u, v) = (\gamma v + \mu)\mathbf{b}(u) + (\alpha v + \beta)\mathbf{d}(u),$$

where α, β, γ, and μ are real constants with $\delta \equiv \alpha\mu - \beta\gamma \neq 0$. Let $s(u)$ and $\bar{s}(u)$ be the discriminant polynomials resulting from using $\mathbf{q}(u, v)$ and $\bar{\mathbf{q}}(u, v)$, respectively (see Section 1.3). Then $s(u) = \delta^2 \bar{s}(u)$.

Proof. Substituting the two parameterizations $\mathbf{q}(u, v)$ and $\bar{\mathbf{q}}(u, v)$ of $\mathcal{S}$ into $X^{\mathrm{T}}AX = 0$ yields the following two different quadratic equations, in place of (2):

$$c_2(u)v^2 + 2c_1(u)v + c_0(u) = 0,$$

and

$$c_2(u)(\alpha v + \beta)^2 + 2c_1(u)(\alpha v + \beta)(\gamma v + \mu) + c_0(u)(\gamma v + \mu)^2 = 0. \tag{6}$$

Then $s(u) = c_1^2(u) - c_0(u)c_2(u)$ and $\bar{s}(u)$ is the discriminant of the quadratic equation (6) in v. It is straightforward to verify that $s(u) = \delta^2 \bar{s}(u)$. This completes the proof. □

Theorem 4. *If the QSIC is singular with an acnode plus a connected component, then either* (a) *$s(u)$ has a double root u_0 with $s''(u_0) < 0$ and two simple real roots; or* (b) *$s(u) = c_1^2(u)$ where $c_1(u)$ is a quadratic polynomial with two complex conjugate roots. Furthermore, the parameterization $\mathbf{p}(u)$ of the QSIC given by* (4) *is rational if and only if case* (b) *occurs.*

Proof. We first show that, without loss of generality, it may be assumed that either (i) the generating line $\mathbf{b}(u)$ of $\mathcal{S}$ passes through a real regular point N_0 of the QSIC when the parameterization surface $\mathcal{S}$ is doubly ruled; or (ii) the generating conic $\mathbf{d}(u)$ of $\mathcal{S}$ passes through a real regular point N_0 of the QSIC when the parameterization surface $\mathcal{S}$ is singly ruled. For otherwise, if $\mathcal{S}$ is doubly ruled, we may choose an appropriate v_0 such that the line $\bar{\mathbf{b}}(u) = \mathbf{b}(u) + v_0\mathbf{d}(u)$ passes through a real regular point N_0 of the QSIC, and then use the following parameterization for $\mathcal{S}$ (see Fig. 1):

$$\bar{\mathbf{q}}(u,v) = \bar{\mathbf{b}}(u) + v\mathbf{d}(u) = \mathbf{b}(u) + (v + v_0)\mathbf{d}(u).$$

By Lemma 3, the same $s(u)$ will result from using $\bar{\mathbf{q}}(u,v)$ or $\mathbf{q}(u,v)$. If $\mathcal{S}$ is singly ruled and the conic $\mathbf{d}(u)$ does not pass through any real regular point of the QSIC, we may choose an appropriate γ_0 such that the conic $\bar{\mathbf{d}}(u) = \gamma_0\mathbf{b}(u) + \mathbf{d}(u)$ passes through a real regular point N_0 of the QSIC, and then use the following parameterization for $\mathcal{S}$:

$$\bar{\mathbf{q}}(u,v) = \mathbf{b}(u) + v\bar{\mathbf{d}}(u) = (\gamma_0 v + 1)\mathbf{b}(u) + v\mathbf{d}(u).$$

By Lemma 3, again, the same $s(u)$ will result from using $\bar{\mathbf{q}}(u,v)$ or $\mathbf{q}(u,v)$. Thus, in both cases where $\mathcal{S}$ is doubly ruled or singly ruled, the discriminant polynomial $s(u)$ remains the same with or without the assumption made at the start of this proof.

Consider the stereographic projection $\mathbf{M}:\mathcal{S} \to \mathcal{P}$, centered at N_0. Suppose that the QSIC is mapped by $\mathbf{M}$ to the planar cubic $\mathcal{H}$, its acnode X_0 mapped to the acnode $\tilde{X}_0$ of $\mathcal{H}$, and the lines in $\mathcal{L}$ on $\mathcal{S}$ mapped to the pencil of lines centered at a point $\tilde{R}_0$ on the cubic $\mathcal{H}$ in the plane $\mathcal{P}$.

There are two subcases: (a) $\tilde{R}_0$ is a regular point of $\mathcal{H}$ and therefore $\tilde{R}_0 \neq \tilde{X}_0$; and (b) $\tilde{R}_0 = \tilde{X}_0$. In case (a), the line $\tilde{R}_0\tilde{X}_0$ has a double intersection with $\mathcal{H}$ at $\tilde{X}_0$. (See Fig. 2.) Thus $s(u)$ has a multiple root u_0 which gives rise to the line $\tilde{R}_0\tilde{X}_0$. Since $\tilde{X}_0$ is an isolated real point, we have $s(u_0 \pm \varepsilon) < 0$ for a sufficiently small $\varepsilon > 0$. It follows that u_0 is a double root with $s''(u_0) < 0$. Moreover, since two tangents can be drawn from the point $\tilde{R}_0$ to the cubic $\mathcal{H}$ (Salmon, 1934), $s(u)$ has two simple real roots.

In case (b), a line $\tilde{\ell} \in \tilde{\mathcal{L}}$ has two fixed intersection points with the cubic $\mathcal{H}$ at $\tilde{X}_0$ and only one variable intersection point with $\mathcal{H}$, or equivalently, a line $\ell \in \mathcal{L}$ on the ruled quadric $\mathcal{S}$ has, in general, only one variable intersection with the QSIC. This implies that $\mathcal{S}$ is a singly ruled quadric with its vertex at the double point X_0 of the QSIC; for otherwise, if $\mathcal{S}$ was doubly ruled, then a line $\ell \in \mathcal{L}$ would have, in general, two variable intersections with the QSIC (which is irreducible); this is a contradiction. Recall that the two solutions for v of Eq. (2) give the two intersections of ℓ with the QSIC, or equivalently, the two intersections $\tilde{I}_0$ and $\tilde{I}_1$ of $\tilde{\ell}$ with the cubic $\mathcal{H}$. Thus the last coefficient $c_0(u) \equiv 0$, accounting for the

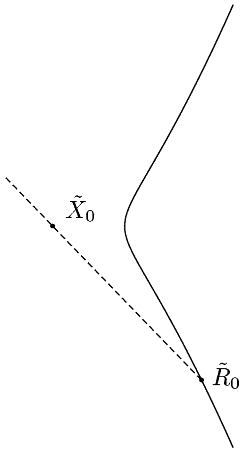

Fig. 2. A singular cubic with an acnode $\tilde{X}_0$ and $\tilde{R}_0 \neq \tilde{X}_0$.

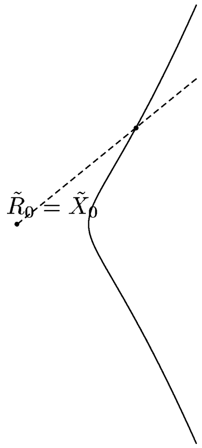

Fig. 3. A singular cubic with an acnode $\tilde{X}_0$ and $\tilde{R}_0 = \tilde{X}_0$.

fact that $v = 0$ is always a solution of Eq. (2), i.e., one of $\tilde{I}_0$ and $\tilde{I}_1$ is always at $\tilde{R}_0$ and the other is a variable intersection given by $v = -2c_1(u)/c_2(u)$. So

$$s(u) = c_1^2(u) - 4c_0(u)c_2(u) = c_1^2(u).$$

Furthermore, since $\tilde{R}_0 = \tilde{X}_0$ is an isolated singular point, the only variable intersection point between the pencil $\tilde{\mathcal{L}}$ and the cubic $\mathcal{H}$ cannot be at $\tilde{R}_0$ for any real value of u. (See Fig. 3.) It follows that $v = -2c_1(u)/c_2(u)$ does not vanish for any real value of u. Hence, $c_1(u)$ has two complex conjugate roots. In this case no real tangent can be drawn from $\tilde{R}_0 = \tilde{X}_0$ to $\mathcal{H}$ and the parameterization (4) becomes rational, given by $\mathbf{p}(u) = c_2(u)\mathbf{b}(u) - 2c_1\mathbf{d}(u)$. This completes the proof. □

Remark. When counting the tangents that can be drawn from $\tilde{R}_0$ to $\mathcal{H}$, we are concerned only with whether or not the two variable intersection points $\tilde{I}_0$ and $\tilde{I}_1$ between a line $\tilde{\ell} \in \tilde{\mathcal{L}}$ and $\mathcal{H}$ coincide. Therefore, unless $\tilde{R}_0$ is an inflection point of $\mathcal{H}$, the tangent of $\mathcal{H}$ at $\tilde{R}_0$ is not counted, since in this case $\tilde{I}_0$ and $\tilde{I}_1$ are distinct and only one of them coincides with $\tilde{R}_0$. This convention on counting the number of tangents from $\tilde{R}_0$ to $\mathcal{H}$ is followed throughout this section.

We can also prove the following results regarding a QSIC with a crunode or a cusp. The proofs are similar to the proof of Theorem 4, so are omitted.

Theorem 5. *If the QSIC is singular with a crunode, then either* (a) $s(u)$ *has a double root* u_0 *with* $s''(u_0) > 0$ *plus two simple real roots*; *or* (b) $s(u) = c_1^2(u)$ *where* $c_1(u)$ *has two distinct real roots. Furthermore, the parameterization* $\mathbf{p}(u)$ *of the QSIC given by* (4) *is rational if and only if case* (b) *occurs.*

Remark. In case (a) the double root of $s(u)$ is generated by a line drawn from a regular point $\tilde{R}_0$ on $\mathcal{H}$ to its crunode $\tilde{X}_0$, and the other two simple roots correspond to two real tangents that can be drawn from $\tilde{R}_0$ to $\mathcal{H}$ (Salmon, 1934). (See Fig. 4.) In case (b), the two simple roots of $c_1(u)$ correspond to the two tangents of $\mathcal{H}$ at its crunode. (See Fig. 5.)

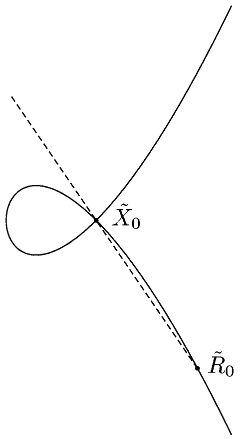

Fig. 4. A singular cubic with a crunode $\tilde{X}_0$ and $\tilde{R}_0 \neq \tilde{X}_0$.

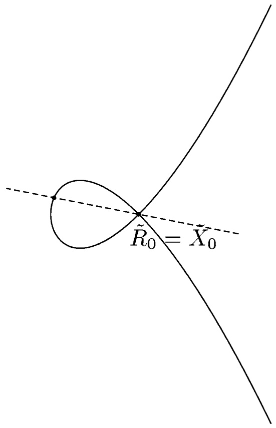

Fig. 5. A singular cubic with a crunode $\tilde{X}_0$ and $\tilde{R}_0 = \tilde{X}_0$.

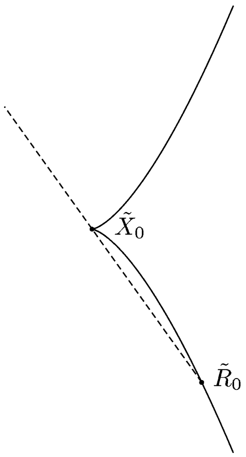

Fig. 6. A singular cubic with a cusp $\tilde{X}_0$ and $\tilde{R}_0 \neq \tilde{X}_0$.

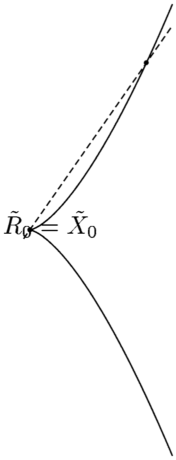

Fig. 7. A singular cubic with a cusp $\tilde{X}_0$ and $\tilde{R}_0 = \tilde{X}_0$.

Theorem 6. *If the QSIC is singular with a cusp, then either* (a) $s(u)$ *has a triple root* u_0 *plus another simple real root*; *or* (b) $s(u) = c_1^2(u)$, *where* $c_1(u)$ *has a real double root. Furthermore, the parameterization* $\mathbf{p}(u)$ *of the QSIC given by* (4) *is rational if and only if case* (b) *occurs.*

Remark. In case (a) the triple root of $s(u)$ is generated by a line drawn from a regular point $\tilde{R}_0$ on $\mathcal{H}$ to its cusp, and the simple root of $s(u)$ corresponds to a real tangent that can be drawn from $\tilde{R}_0$ to $\mathcal{H}$ (Salmon, 1934). (See Fig. 6.) In case (b), the double root of $c_1(u)$ corresponds to the unique tangent of $\mathcal{H}$ at its cusp. (See Fig. 7.)

By Theorems 4–6, in particular the argument in the proof for case (b) of Theorem 4, we obtain the following theorem.

Theorem 7. *For a QSIC with one singular point* X_0, *the parameterization* $\mathbf{p}(u)$ *by* (4) *is rational if and only if the parameterization surface* $\mathcal{S}$ *used in Levin's method is a singly ruled quadric with its vertex at* X_0. *Furthermore, when such a parameterization surface is used, a rational parameterization of the QSIC is given by*

$$\mathbf{p}(u) = c_2(u)\mathbf{b}(u) - 2c_1\mathbf{d}(u).$$

Theorem 8. *Let* $\mathcal{A}$: $X^{\mathrm{T}}AX = 0$ *and* $\mathcal{B}$: $X^{\mathrm{T}}BX = 0$ *be two quadrics whose intersection curve is singular but nonplanar. Then there exists a singly ruled quadric in the pencil of* $\mathcal{A}$ *and* $\mathcal{B}$ *whose vertex is at a singular point of the intersection curve of* $\mathcal{A}$ *and* $\mathcal{B}$.

Proof. Let X_0 be a singular point of the QSIC of $\mathcal{A}$ and $\mathcal{B}$. Suppose that neither $\mathcal{A}$ nor $\mathcal{B}$ is singly ruled with its vertex at X_0, for otherwise we are done. Then the respective tangent planes $X_0^{\mathrm{T}}AX = 0$ and $X_0^{\mathrm{T}}BX = 0$ of $\mathcal{A}$ and $\mathcal{B}$ at X_0 are well-defined and identical, i.e., $X_0^{\mathrm{T}}A = \rho X_0^{\mathrm{T}}B$ for some real constant ρ. Hence, the quadric $\mathcal{S}_0$: $X^{\mathrm{T}}(A - \rho B)X = 0$ in the pencil of $\mathcal{A}$ or $\mathcal{B}$ is a singly ruled quadric with its vertex at X_0, since $X_0^{\mathrm{T}}(A - \rho B) = 0$. Note that $\mathcal{S}_0$ cannot be a pair of planes, since the QSIC is nonplanar. This completes the proof. □

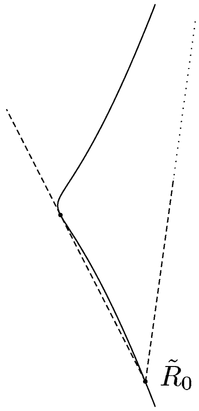

Fig. 8. A nonsingular cubic with one component.

Note that Theorem 8 also covers the case where a QSIC consists of a line and a space cubic.

Corollary 9. *Any singular QSIC is rational.*

3.3. Nonsingular intersection curves

Theorem 10. *If the QSIC is nonsingular with one connected component, then $s(u)$ has two simple real roots and two complex conjugate roots.*

Proof. Similar to the proof of Theorem 4, with the aid of the stereographic projection **M**, the argument is reduced to counting the number of real tangents that can be drawn to a nonplanar cubic $\mathcal{H}$ from a regular point $\tilde{R}_0$ on $\mathcal{H}$, since each of these tangents is accounted for by a simple real root of $s(u)$. Since $\mathcal{H}$ has one connected component, we know that exactly two tangents can be drawn from $\tilde{R}_0$ to $\mathcal{H}$ (Salmon, 1934). (See Fig. 8.) Hence, $s(u)$ has two simple real roots and two other complex conjugate roots. This completes the proof. □

Theorem 11. *If the QSIC is nonsingular with two connected components, then either* (a) *$s(u)$ has four simple real roots*; *or* (b) *$s(u)$ has no real roots and $s(u) > 0$ for all u.*

The proof of Theorem 11 is similar to that of Theorem 10, so is omitted. The two cases in Theorem 11 correspond to whether the pencil center $\tilde{R}_0$ is on the infinite branch or the oval branch of the cubic $\mathcal{H}$. When $\tilde{R}_0$ is on the infinite branch, four tangents can be drawn from $\tilde{R}_0$ to $\mathcal{H}$ (see Fig. 9); when $\tilde{R}_0$ is on the oval branch, no tangent can be drawn from $\tilde{R}_0$ to $\mathcal{H}$ (Salmon, 1934) (see Fig. 10).

3.4. Complete characterization of irreducible QSIC's

In Sections 3.2 and 3.3 we obtained necessary conditions in terms of the roots of $s(u)$ for all morphologies of irreducible QSIC's. Because these necessary conditions are distinct from each other, they are therefore also sufficient for the respective morphologies. We summarize these necessary and sufficient conditions for different cases in the following theorem.

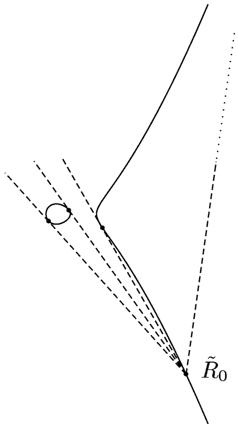

Fig. 9. A nonsingular cubic with two components and $\tilde{R}_0$ on the infinite branch.

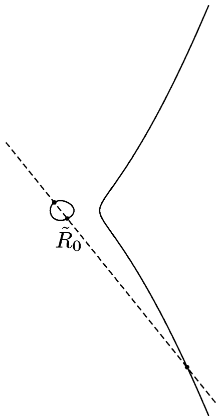

Fig. 10. A nonsingular cubic with two components and $\tilde{R}_0$ on the oval branch.

Theorem 12. *Let $\mathcal{Q}$ be an irreducible QSIC.*

(1) *$\mathcal{Q}$ has one real singular point but no other real points if and only if $s(u) < 0$ for all u except for one value u_0 at which $s(u_0) = 0$.*
(2) *$\mathcal{Q}$ is singular with an acnode plus a connected component if and only if either* (a) *$s(u)$ has a double root u_0 with $s''(u_0) < 0$ and two simple real roots; or* (b) *$s(u) = c_1^2(u)$ where $c_1(u)$ has two complex conjugate roots. The parameterization $\mathbf{p}(u)$ of $\mathcal{Q}$ given by* (4) *is rational in case* (b).
(3) *$\mathcal{Q}$ is singular with a crunode if and only if either* (a) *$s(u)$ has a double root u_0 with $s''(u_0) > 0$ plus two simple real roots; or* (b) *$s(u) = c_1^2(u)$ where $c_1(u)$ has two distinct real roots. The parameterization $\mathbf{p}(u)$ of $\mathcal{Q}$ given by* (4) *is rational in case* (b).
(4) *$\mathcal{Q}$ is singular with a cusp if and only if either* (a) *$s(u)$ has a triple root u_0 plus another simple real root; or* (b) *$s(u) = c_1^2(u)$ where $c_1(u)$ has a real double root. The parameterization $\mathbf{p}(u)$ of $\mathcal{Q}$ given by* (4) *is rational in case* (b).
(5) *$\mathcal{Q}$ is devoid of real points if and only if $s(u) < 0$ for all u.*
(6) *$\mathcal{Q}$ is nonsingular with one connected component if and only if $s(u)$ has two simple real roots and two complex conjugate roots.*
(7) *$\mathcal{Q}$ is nonsingular with two connected components if and only if either* (a) *$s(u)$ has four simple real roots; or* (b) *$s(u)$ has no real roots and $s(u) > 0$ for all u.*

4. Enhanced Levin's method

Based on the procedure in Section 2 and the results summarized in Section 3.4, we are now able to present an enhanced Levin's method, to be referred to as ELM, that is capable of detecting reducibility and singularity, as well as counting the connected components of a QSIC. In addition, the method produces a rational parameterization for a singular QSIC through a special selection of the parameterization surface.

Procedure ELM
Input: Two quadrics $\mathcal{A}$: $X^{\mathrm{T}}AX = 0$ and $\mathcal{B}$: $X^{\mathrm{T}}BX = 0$.
Output: A parameterization $\mathbf{p}(u)$ given by (4) of the QSIC of $\mathcal{A}$ and $\mathcal{B}$, together with its reducibility, the type of singularity, or the number of connected components. $\mathbf{p}(u)$ is rational if the QSIC is singular.

Begin

(1) (*For a planar QSIC.*) Detect if there is a pair of planes in the pencil of $\mathcal{A}$ and $\mathcal{B}$. If yes, compute the planar QSIC, and quit; otherwise go to step (2).
(2) (*For a QSIC consisting of a line ℓ_1 and a space cubic $\mathcal{C}$.*) Find a real ruled quadric $\mathcal{S}$ in the pencil of $\mathcal{A}$ and $\mathcal{B}$. Use the resultant-based procedure in Section 2.2 to detect whether the QSIC has a linear component. If not, go to step (3). If yes, extract the linear component ℓ_1, and then go to step (3) to compute a rational parameterization of the remaining cubic component of the QSIC.
(3) (*For a singular QSIC with one real singular point or the cubic component of a nonplanar reducible QSIC.*) Find all singly ruled quadrics $\mathcal{S}$ in the pencil of $\mathcal{A}$ and $\mathcal{B}$. If there is no such surface, go to step (4). For each singly ruled surface in the pencil, compute its vertex and check whether or not the vertex is on $\mathcal{A}$. If any of the singly ruled quadrics, say $\mathcal{S}_0$, has its vertex X_0 on $\mathcal{A}$, then X_0 is a singular point of the QSIC, and the parameterization $\mathbf{p}(u)$ of the QSIC constructed with $\mathcal{S}_0$ as the parameterization surface is rational (by Theorem 7). If none of the singly ruled quadrics in the pencil has its vertex on $\mathcal{A}$, go to step (4).
(4) (*For a nonsingular QSIC.*) Use any real ruled quadric $\mathcal{S}$ in the pencil of $\mathcal{A}$ and $\mathcal{B}$ to generate a parameterization $\mathbf{p}(u)$ of the QSIC by (4). Use the conditions in Theorem 12 to compute the number of connected components of the QSIC.

End

Finding a pair of planes or a parameterization surface $\mathcal{S}$ in a quadric pencil $X^{\mathrm{T}}(\lambda A + B)X = 0$ entails solving for the roots of the quartic equation $|\lambda A + B| = 0$, as well as determining the multiplicities of these roots. The need for solving or analyzing a quartic equation also arises from analyzing the discriminant polynomial $s(u)$ (3), as required in ELM. We will not discuss here in detail techniques for solving quartic equations, but instead refer the interested reader to (Dickson, 1914; Uspensky, 1948).

The following three examples of computing QSIC's using ELM verify some of the conditions listed in Theorem 12.

Example 1. (*Reducible intersection of two cones, comprising a line and a space cubic. See Fig.* 11.) The matrices of the input quadrics are

$$A = \begin{bmatrix} 1.0 & 0.0 & 0.0 & -0.5 \\ 0.0 & 0.75 & -0.5 & -0.5 \\ 0.0 & -0.5 & 0.0 & 0.0 \\ -0.5 & -0.5 & 0.0 & 0.25 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.75 & 0.0 & -0.5 & 0.125 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ -0.5 & 0.0 & 0.0 & 0.25 \\ 0.125 & 0.0 & 0.25 & -0.3125 \end{bmatrix}.$$

Using ELM, the parameterization surface $\mathcal{S}$ is a cone (see Fig. 12), and

$$s(u) = \left(u^2 - 1\right)^2.$$

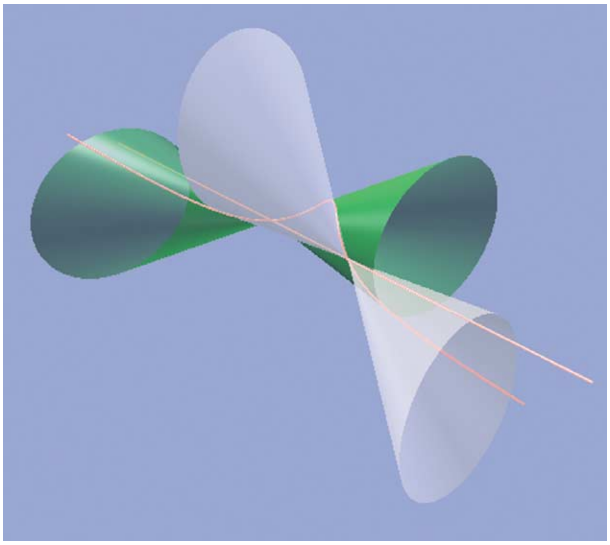

Fig. 11. The reducible intersection of two cones.

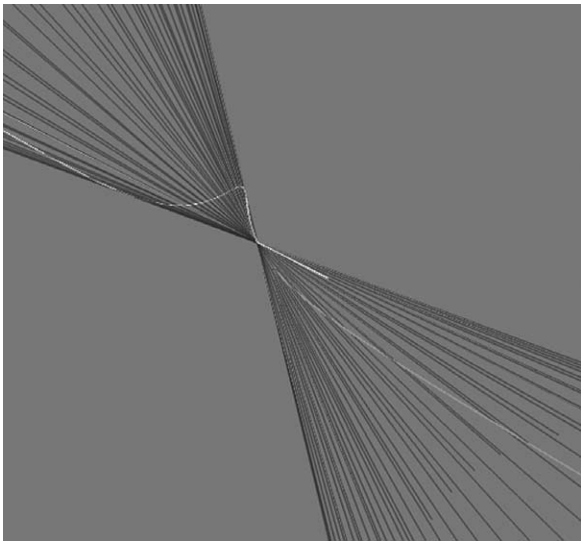

Fig. 12. The same intersection curve as in Fig. 11 with the parameterization surface $\mathcal{S}$ derived by ELM.

The roots of $s(u)$ are: $u_1 = 1.0$, $u_2 = 1.0$, $u_3 = -1.0$, $u_4 = -1.0$. The linear component is

$$(0.5, 0, -1, 1)^{\mathrm{T}} + v(0, 0, 3.46, 0)^{\mathrm{T}}.$$

A rational parameterization of the cubic component is

$$\mathbf{p}(u) = \begin{pmatrix} 0.09u^3 - 2.76u^2 - 4.49u + 1.82 \\ -6.92u^3 + 6.92u^2 - 6.92u + 2.3 \\ 0.36u^3 + 4.97u^2 + 1.53u + 3.81 \\ -3.82u^3 - 1.51u^2 - 4.99u - 0.34 \end{pmatrix}.$$

Example 2. (*Singular intersection with a cusp of a sphere and a cone. See Fig.* 13.) The matrices of the input quadrics are

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using ELM, the parameterization surface $\mathcal{S}$ is a cone (see Fig. 14), and

$$s(u) = (u - 1)^4.$$

The roots of $s(u)$ are: $u_1 = 1.0$, $u_2 = 1.0$, $u_3 = 1.0$, $u_4 = 1.0$ (ref. condition 4(b) in Theorem 12). A rational parameterization of the QSIC is

$$\mathbf{p}(u) = \begin{pmatrix} -1.414u^4 + 2.828u^3 - 2.828u + 1.414 \\ -u^4 + 4.0u^3 - 6.0u^2 + 4.0u - 1.0 \\ u^4 - 2u^2 + 1 \\ -2u^4 - 4u^2 - 2 \end{pmatrix}.$$

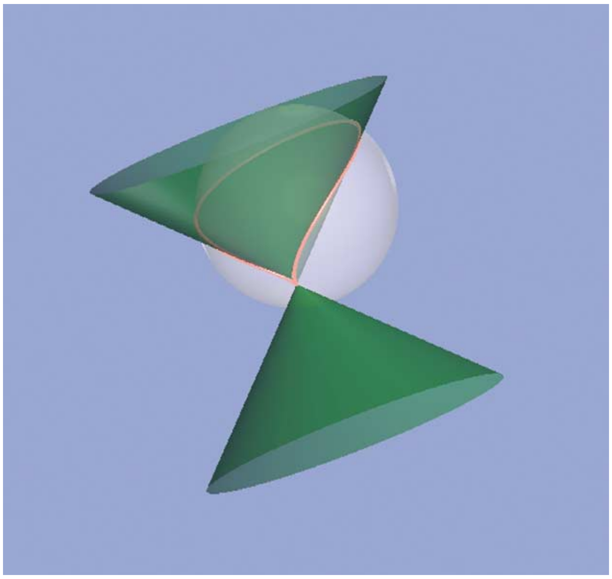

Fig. 13. The singular intersection of a sphere and a cone with a cusp.

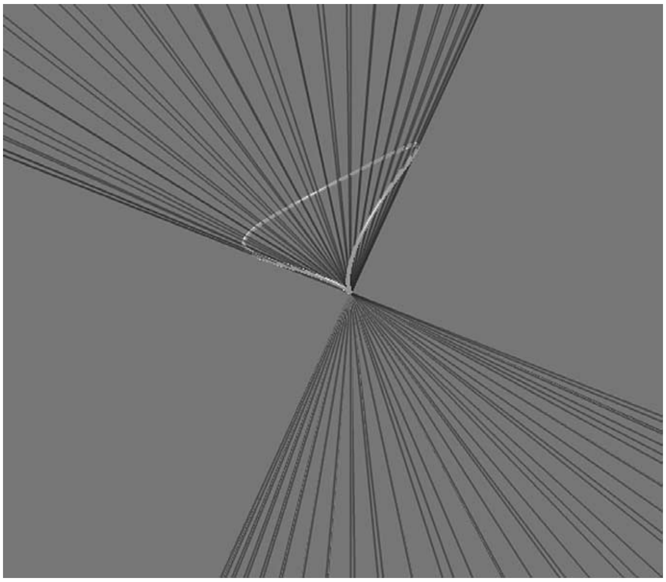

Fig. 14. The same intersection curve as in Fig. 13 with the parameterization surface $\mathcal{S}$ derived by ELM.

Example 3. (*Nonsingular intersection of an ellipsoid and a two-sheeted hyperboloid with two components. See Figures* 15 *and* 16.) The matrices of the input quadrics are

$$A = \begin{bmatrix} 3.993 & -0.448 & -2.606 & 0.0 \\ -0.448 & -3.381 & -3.356 & 0.0 \\ -2.606 & -3.356 & 4.177 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2.778 & 0.008 & 0.050 & 0.528 \\ 0.008 & 2.662 & 0.047 & -0.764 \\ 0.050 & 0.047 & 2.847 & 0.972 \\ 0.528 & -0.764 & 0.972 & -0.845 \end{bmatrix}.$$

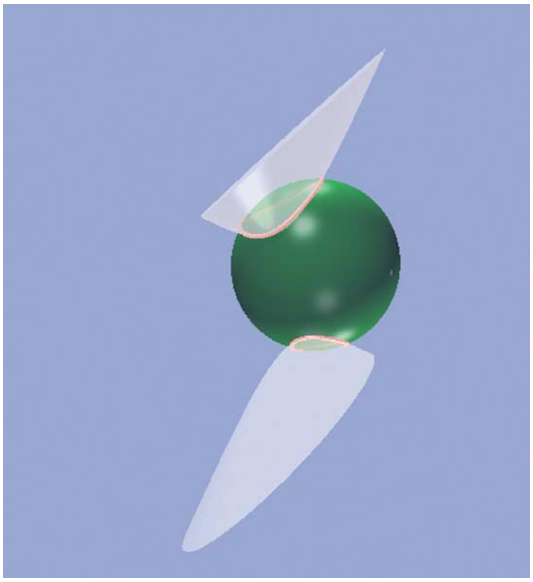

Fig. 15. The nonsingular intersection of an ellipsoid and a two-sheeted hyperboloid with two components.

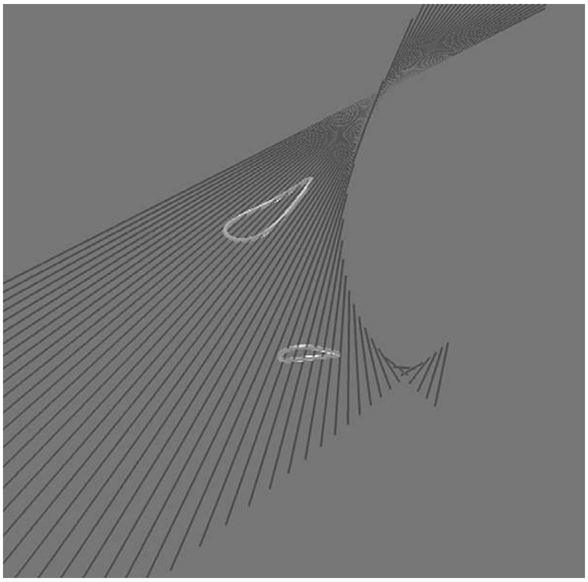

Fig. 16. The same intersection curve as in Fig. 15 with the parameterization surface $\mathcal{S}$.

Using ELM, the parameterization surface $\mathcal{S}$ is a hyperbolic paraboloid, and

$$s(u) = -2.39u^4 - 2.41u^3 + 3.48u^2 + 3.17u - 0.14.$$

The roots of $s(u)$ are:

$$u_1 = -1.324, \qquad u_2 = -0.897, \qquad u_3 = 0.043, \qquad u_4 = 1.169$$

(ref. condition 7(a) in Theorem 12). A parameterization of the QSIC is

$$\mathbf{p}(u) = \begin{pmatrix} 0.84u^3 + 0.19u^2 - 0.62u - 0.09 \\ 1.11u^3 + 2.11u^2 + 1.72u + 0.39 \\ -0.53u^3 - 1.15u^2 + 0.09u + 0.35 \\ 2.51u^2 + 2.54u + 0.88 \end{pmatrix} \pm \sqrt{s(u)} \begin{pmatrix} -0.74u - 0.49 \\ 0.34u + 0.29 \\ -0.48u \\ 0 \end{pmatrix}.$$

5. Conclusions

We have presented an analysis of Levin's method for computing the intersection curve of two quadric surfaces (QSIC). We have introduced additional tests in order to make this method capable of computing geometric and structural information—irreducibility, singularity, and the number of connected components—for the QSIC. We have also provided an enhanced version of Levin's method that generates a rational parameterization for any singular QSIC. Further research is still required to examine the numerical accuracy of Levin's method in order to insure that this method is numerically robust.

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Appendix A

Levin' method is based on the existence of a real ruled quadric in any quadric pencil. The proof given by Levin (1976) for this result is involved and lengthy, filling nearly two double-columned pages. In this appendix we present a geometric proof, which is not only shorter, but also provides useful insight into this geometric fact.

Theorem A.1. *There exists a real ruled quadric in the pencil of any two distinct real quadrics.*

Proof. Let $\mathcal{A}$: $X^{\mathrm{T}}AX = 0$ and $\mathcal{B}$: $X^{\mathrm{T}}BX = 0$ be two distinct quadrics, with *real* coefficient matrices A and B. Let X_0 and X_1 be two distinct real points on the QSIC of $\mathcal{A}$ and $\mathcal{B}$, then the line passing through X_0 and X_1, denoted by X_0X_1, is a real line. If there do not exist two real points on the QSIC, then we choose X_0 and X_1 to be two complex conjugate points on the QSIC. We now show that the line X_0X_1 is also real. Let $X_0 = U + \mathrm{i}V$ and $X_1 = U - \mathrm{i}V$, where U and V are two linear independent real 4D vectors (i.e., two distinct real points). Then $U = (X_0 + X_1)/2$ and $V = (X_0 - X_1)/(2\mathrm{i})$, as linear combinations of X_0 and X_1, are two distinct real points on the line X_0X_1. Hence, X_0X_1 is a real line.

Next we choose a real point X^* on the line X_0X_1 such that X^* is distinct from X_0 or X_1. Obviously there exists λ_0 such that X^* is on the quadric $\mathcal{S}$: $X^{\mathrm{T}}SX \equiv X^{\mathrm{T}}(\lambda_0 A + B)X = 0$; note that, if X^* is also on the QSIC, i.e., X^* is on both $\mathcal{A}$ and $\mathcal{B}$, then X^* is on $X^{\mathrm{T}}(\lambda A + B)X = 0$ for any λ. Hence, by Bézout's theorem, $\mathcal{S}$ contains the real line X_0X_1 since it contains three distinct points X_0, X_1, and X^* on the line. It follows that $\mathcal{S}$ is a real ruled quadric, for otherwise $\mathcal{S}$ would be an ellipsoid, an elliptic paraboloid, or a two-sheet hyperboloid, which could not contain any real line. This completes the proof. □

Although Levin's method works as long as there exists a real ruled quadric in the pencil of two input quadrics, one might prefer to use some special ruled quadrics for the parameterization surface $\mathcal{S}$ that have relatively simple parameterizations in affine space; this viewpoint is espoused in Levin's paper (1976). For this reason, the ruled quadric $\mathcal{S}$ that is allowed as the parameterization surface in Levin's original method can only be a pair of planes, a hyperbolic paraboloid, or a cylinder (Levin, 1976, p. 560). (Incidentally, Levin adds the cone into the list of allowable parameterization surfaces in his other paper (Levin, 1978, Table 1, p. 75).) The next theorem is the specific result stated and proved by Levin, for which we now give a more concise proof.

Theorem A.2 (Levin, 1976, p. 561). *The intersection curve of two quadric surfaces lies in a plane, pair of planes, hyperbolic or parabolic cylinder, or a hyperbolic paraboloid.*

Proof. Let $\mathcal{A}$: $X^{\mathrm{T}}AX = 0$ and $\mathcal{B}$: $X^{\mathrm{T}}BX = 0$ be two distinct quadrics, where $X = (x, y, z, w)^{\mathrm{T}}$ are homogeneous coordinates with $w = 0$ representing the plane at infinity. Let the upper 3×3 submatrices of A and B be denoted by A_u and B_u, respectively. The intersection of the pencil of $X^{\mathrm{T}}(\lambda A + B)X = 0$ with the plane at infinity $w = 0$ is the pencil of conics $\bar{X}^{\mathrm{T}}(\lambda A_u + B_u)\bar{X} = 0$, where $\bar{X} = (x, y, z)$. We may assume that the conics $\bar{\mathcal{A}}$: $\bar{X}^{\mathrm{T}} A_u \bar{X} = 0$ and $\bar{\mathcal{B}}$: $\bar{X}^{\mathrm{T}} B_u \bar{X} = 0$ are distinct, for otherwise the QSIC of $\mathcal{A}$ and $\mathcal{B}$ is planar and the proof is done. Clearly, all the special quadrics $X^{\mathrm{T}} S X = 0$ listed in Theorem A.2 are characterized by the fact that their intersection with the plane $w = 0$ is a degenerate conic containing a real line.

Now the proof proceeds with much the same idea as the proof of Theorem A.1; we need to find a conic in the pencil $\bar{X}^{\mathrm{T}}(\lambda A_u + B_u)\bar{X} = 0$ that contains a real line. First choose two points $\bar{X}_0$ and $\bar{X}_1$ to be either (1) two distinct real intersection points of $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$; or (2) two distinct complex conjugate intersection points of $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$; or (3) both at the only intersection point $\bar{I}_0$ of $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ if $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ intersect at $\bar{I}_0$ with multiplicity 4. In cases (1) and (2), we have a unique real line $\bar{X}_0\bar{X}_1$ connecting $\bar{X}_0$ and $\bar{X}_1$. In case (3) there exists a line that has at least double contact with both $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ at $\bar{I}_0$ and thus also has at least double contact at $\bar{I}_0$ with any conic in the pencil of $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$; in this case the line is still denoted by $\bar{X}_0\bar{X}_1$ for notational uniformity. Now choose another real point $\bar{X}^*$ on the line $\bar{X}_0\bar{X}_1$ that is distinct from $\bar{X}_0$ or $\bar{X}_1$. Then there exists λ_0 such that the conic $\bar{X}^{\mathrm{T}}(\lambda_0 A_u + B_u)\bar{X} = 0$ contains $\bar{X}^*$. It follows, by Bézout's theorem, that the conic $\bar{X}^{\mathrm{T}}(\lambda_0 A_u + B_u)\bar{X} = 0$ contains the real line $\bar{X}_0\bar{X}_1$, since it contains $\bar{X}^*$ and two other points $\bar{X}_0$ and $\bar{X}_1$ (with multiplicity counted) on the line. Hence, the QSIC of $\mathcal{A}$ and $\mathcal{B}$ lies on the corresponding quadric $X^{\mathrm{T}}(\lambda_0 A + B)X = 0$, which is one of the special quadrics listed in Theorem A.2. This completes the proof. □

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