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An algebraic approach to continuous collision detection for ellipsoids ☆

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ABSTRACT

We present algebraic expressions for characterizing three configurations formed by two ellipsoids in $\mathbb{R}^3$ that are relevant to collision detection: separation, external touching and overlapping. These conditions are given in terms of explicit formulae expressed by the subresultant sequence of the characteristic polynomial of the two ellipsoids and its derivative. For any two ellipsoids, the signs of these formulae can easily be evaluated to classify their configuration. Furthermore, based on these algebraic conditions, an efficient method is developed for continuous collision detection of two moving ellipsoids under arbitrary motions.

1. Introduction

Collision detection finds many applications in computer graphics, computer animation, CAD/CAM as well as computational physics (see Eberly, 2001). Since collision detection for general free-form moving objects is computationally very expensive, bounding volumes are often used to approximate the free-form objects in order to reduce the computational cost. Ellipsoids are good candidates for such bounding volumes since they have low algebraic degree and are tight bounding volumes for a wide class of objects (Bouville, 1985; Lu et al., 2007). Minimum bounding (or enclosing) ellipsoids and their computations have long been studied as a classical mathematical problem, see e.g., Welzl (1991), Kumar and Yildirim (2005), Todd and Yildirim (2007), Schröker (2008), and have important applications to not only CAGD and computer graphics but also other areas such as data uncertainty analysis.

Collision detection for ellipsoids has been an active research topic over the past years, see Ju et al. (2001), Rimon and Boyd (1997), Shiang et al. (2000), Wang et al. (2004), Choi et al. (2006, 2009). Wang et al. (2001) provide algebraic conditions for the three important configurations, i.e., *separation*, *external touching* and *overlapping*, of two static ellipsoids in $\mathbb{R}^3$, the three dimensional affine space. It is shown that the relative position of two ellipsoids is related to the root pattern of their characteristic equation. Specifically, two ellipsoids are separate if and only if their characteristic equation has two distinct positive roots; and they touch each other externally if and only if the characteristic equation has one positive double root; and otherwise they overlap. This result allows us to determine the configuration of two ellipsoids by simply counting the number of positive roots of the characteristic polynomial. The algebraic conditions given by Wang et al. (2001) lays the theoretical foundation for the follow-up practical applications in collision detection for ellipsoids.

Collision detection mostly deals with moving objects. When the positions of the two ellipsoids are given in a sequence of discrete time frames, which is often the situation in computer animation, temporal and geometric coherence are exploited

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in Wang et al. (2004) for speeding up the computations. A separating plane for two non-intersecting ellipsoids at a frame is calculated to help quickly identify whether the ellipsoids are still separate or not in the next frame. A more computationally intensive ellipsoid–ellipsoid intersection test needs to be carried out only when the separating plane fails to guarantee the separation of the ellipsoids.

Continuous collision detection (CCD) in which object motions are given by continuous functions of a time parameter t has been gaining increasing interests in the past decade (see e.g., Redon et al., 2002; Teschner et al., 2005). It focuses on determining the collision status of objects over a specific time span and is exact in the sense that no discretization of the time domain is needed. When comes to the CCD of two moving ellipsoids with continuous motions, the method in Wang et al. (2004) mentioned above cannot easily be extended since the computation of a separating plane requires solving for the roots of the characteristic equation. Based on the algebraic condition by Wang et al. (2001), Choi et al. (2003) reduces the problem to analyzing the zero set of the bivariate characteristic polynomial equation formed by the two moving ellipsoids; however, their algorithm involves the brute-force computation of the zero set and is therefore slow. Later Choi et al. (2009) further develop an efficient search scheme to determine the collision time instants from the bivariate characteristic equation in real time. The basic idea is to find the contact time instants by alternative searches in the two parameter domains of the characteristic equation. Unfortunately, the use of Bézier clipping in the search makes the method only applicable to ellipsoids moving under rational motions.

In this paper, we aim at establishing a symbolic approach to determine the relative position of two ellipsoids, which can then be applied to the CCD of two moving ellipsoids under arbitrary motions (such as the commonly used helical motions as in Example 5.2). Our theoretical background is still Wang et al. (2001), but with the difference that we count the number of positive roots of the characteristic polynomial symbolically without resorting to solving for the roots of the characteristic polynomial. This symbolical approach is derived from the classical Descarte's rule and the modified sign variation number of the signed subresultant sequence (e.g., see Basu et al., 2006), and requires only the computation of five explicit formulae from the subresultant sequences of the characteristic polynomial and its derivative. We thereafter develop an algorithm for CCD of two moving ellipsoids under arbitrary continuous motions which are not necessarily rational. In CCD case, the five explicit formulae directly lead to five functions in the time parameter t, and CCD can simply be done by solving these five functions.

We note that Gonzalez-Vega and Mainar (2008) translate the algebraic conditions in Wang et al. (2001) to a set of closed form formulae to characterize the separation of two ellipsoids, and their results are also based on the subresultant sequence of the characteristic polynomial and its first derivative. However, their conditions do not distinguish the two conditions of external touching and overlapping. Our present work improves Gonzalez-Vega and Mainar (2008) by further distinguishing external touching and overlapping, which is significant in collision detection.

The remainder of the paper is organized as follows. In Section 2 we review the algebraic conditions given by Wang et al. (2001) and present the algebraic tool of subresultant sequences which will later be used to derive our explicit formulae. We then derive in Section 3 the five explicit formulae for distinguishing the root patterns of the characteristic equation, thus characterizing the configurations of two ellipsoids. In Section 4, we examine the computational cost by optimizing the evaluation of the five explicit formulae. In Section 5, we present examples on applying our method to continuous collision detection for ellipsoids and compare the efficiency of our approach with Choi et al. (2009). We conclude the paper in Section 6.

2. Preliminaries

Given two ellipsoids $\mathcal{A}$: $X^T AX = 0$ and $\mathcal{B}$: $X^T BX = 0$, where $X = (x, y, z, w)^T$ are the homogeneous coordinates of points $(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}) \in \mathbb{R}^3$ and A, B are 4×4 coefficient matrices with elements in $\mathbb{R}$, the *characteristic polynomial* of the ellipsoids $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\tilde{f}(\lambda) = \det(\lambda A + B).$$

The characteristic polynomial $\tilde{f}(\lambda)$ has degree 4 in $\mathbb{R}[\lambda]$, where $\mathbb{R}[\lambda]$ is the polynomial ring with real coefficients. We define the normalization of $\tilde{f}(\lambda)$ by $f(\lambda) = \tilde{f}(\lambda)/\det(A)$, which can be written as

$$f(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d$$

with $a, b, c, d \in \mathbb{R}$. In the following we also call $f(\lambda)$ the characteristic polynomial. Since A and B represent ellipsoids, we have $\det(A) < 0$ and $\det(B) < 0$; hence $d = \det(B)/\det(A) > 0$. It then follows that zero cannot be a root of $f(\lambda) = 0$. We further assume that the interiors of the ellipsoids $\mathcal{A}$ and $\mathcal{B}$ are defined by $X^T AX < 0$ and $X^T BX < 0$.

Now we define the three configurations of two ellipsoids: separate, external touching and overlapping. An ellipsoid is regarded as a solid bounded by the boundary surface $X^T AX = 0$. Two ellipsoids are *separate* if their boundary surfaces and interiors share no common points; otherwise, they are said to intersect. Furthermore, two intersecting ellipsoids are said to *overlap* if their interiors have a common point; otherwise they *touch externally*. That is, two intersecting ellipsoids may overlap or touch externally.

The following algebraic conditions are given by Wang et al. (2001) on the configurations of two ellipsoids $\mathcal{A}$ and $\mathcal{B}$.

Theorem 2.1. *(See Wang et al., 2001.)*

1. *The characteristic equation* $f(\lambda) = 0$ *always has at least two negative roots*;
2. *The two ellipsoids* $\mathcal{A}$ *and* $\mathcal{B}$ *are separate if and only if* $f(\lambda) = 0$ *has two distinct positive roots*;
3. *The two ellipsoids* $\mathcal{A}$ *and* $\mathcal{B}$ *touch each other externally if and only if* $f(\lambda) = 0$ *has a positive double root*;
4. *The two ellipsoids* $\mathcal{A}$ *and* $\mathcal{B}$ *overlap if and only if* $f(\lambda) = 0$ *has no positive root.*

We shall derive explicit expressions to symbolically determine the root pattern as described in Theorem 2.1. Such a symbolic approach avoids solving for the roots of the characteristic equation and can then be applied for continuous collision of two moving ellipsoids when there is a time parameter involved. Note that a symbolic treatment has been proposed for two ellipses in Choi et al. (2006), which provides a basis for continuous collision detection for two moving ellipses therein. However, it is more difficult for ellipsoids because, unlike the case of ellipses, the appearance of a double root of the characteristic equation does not necessarily mean any configuration change for two ellipsoid. See Choi et al. (2006) for a brief discussion about this difficulty.

We now introduce the concept of the subresultant sequence, an algebraic tool to be used in our derivation. For more details, see for example, Geddes et al. (1992), Basu et al. (2006), von zur Gathen and Gerhard (1999), Kerber (2009).

Definition 2.2. (See Basu et al., 2006.) Let

$$f(\lambda) = \sum_{k=0}^{n} p_k \lambda^k, \qquad g(\lambda) = \sum_{k=0}^{m} q_k \lambda^k$$

be two polynomials in $\mathbb{R}[\lambda]$ with degrees $n = \deg(f) \geqslant \deg(g) = m$. The i-th Sylvester–Habicht matrix of f and g, denoted by $\mathrm{SyHa}_i(f, g)$, is defined by

$$\mathrm{SyHa}_i(f,g) := \begin{pmatrix} p_n & \cdots & \cdots & \cdots & \cdots & p_0 & 0 & 0 \\ 0 & \ddots & & & & & \ddots & 0 \\ \vdots & \ddots & p_n & \cdots & \cdots & \cdots & \cdots & p_0 \\ \vdots & & 0 & q_m & \cdots & \cdots & \cdots & q_0 \\ \vdots & \iddots & \iddots & & & & \iddots & 0 \\ 0 & \iddots & & & & \iddots & \iddots & \vdots \\ q_m & \cdots & \cdots & \cdots & q_0 & 0 & \cdots & 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} m-i \\ \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} n-i. \end{matrix}$$

The i-th signed subresultant $\mathbf{sr}_i(\lambda)$ of f and g is the determinant of the $(n+m-2i) \times (n+m-2i)$ matrix whose first $n+m-2i-1$ columns are taken from the first $n+m-2i-1$ columns of $\mathrm{SyHa}_i(f, g)$, and the last column is the polynomial sequence $\lambda^{n-i-1} f, \lambda^{n-i-2} f, \ldots, f, g, \lambda g, \ldots, \lambda^{m-i-1} g$. Note that the first signed subresultant $\mathbf{sr}_0(\lambda)$ is equal to the resultant $\mathrm{Res}(f, g)$ of f and g.

In the sequel, we consider the signed subresultant sequence of a polynomial f and its first derivative f'. Denote the coefficient of the degree j term of the polynomial $\mathbf{sr}_i$ by $\mathbf{sr}_{ij}$, $j = 0, \ldots, \deg(\mathbf{sr}_i)$. The following property of signed subresultant sequence will be important to our later analysis.

Lemma 2.3. *Let* $f(\lambda) \in \mathbb{R}[x]$. *Then* $\deg(\gcd(f(\lambda), f'(\lambda))) = k$ *if and only if* $\mathbf{sr}_{kk} \neq 0$ *and* $\mathbf{sr}_{ii} \equiv 0$, $i = 0, \ldots, k-1$. *Furthermore, we have* $\mathbf{sr}_k(\lambda) = \gcd(f(\lambda), f'(\lambda))$.

3. Explicit formulae for configurations of two ellipsoids

In this section we shall derive explicit formulae for characterizing the configurations of two static ellipsoids. Throughout we shall repeatedly apply the classical Descartes' rule of signs and the modified sign variations of the signed subresultants. See the details in, for example, Basu et al. (2006).

Proposition 3.1 *(Descartes' rule of signs). Let* $f(\lambda) = a_n\lambda^n + \cdots + a_0 \in \mathbb{R}[\lambda]$. *Then the number of positive roots of* $f(\lambda) = 0$ *is equal to* $\mathrm{Var}(a_n, \ldots, a_0) - 2k$ *for some non-negative integer k, where* $\mathrm{Var}(s)$ *is the number of sign variations in a sequence s.*

Corollary 3.2. *The number of negative roots of* $f(\lambda) = 0$ *is equal to* $\mathrm{Var}((-1)^n a_n, (-1)^{n-1} a_{n-1}, \ldots, a_0) - 2k$ *for some non-negative integer k.*

Table 1
Configurations of two ellipsoids and the corresponding values for $\mathrm{Var}(1,a,b,c,d)$ and the number of real roots for $f(\lambda)=0$.

$\mathrm{Var}(1,a,b,c,d)$	# of real roots for $f(\lambda)=0$	Configuration of two ellipsoids
0	2 or 4	overlap
2	4	separate or externally touch
	2	overlap
4	impossible	

Let $f(\lambda)=\lambda^4+a\lambda^3+b\lambda^2+c\lambda+d$ be the characteristic polynomial of two ellipsoids $\mathcal{A}$ and $\mathcal{B}$. Since $d>0$, $\mathrm{Var}(1,a,b,c,d)$ is even. Then we have the following lemma.

Lemma 3.3. $\mathrm{Var}(1,a,b,c,d)\neq 4$.

Proof. Suppose on the contrary that $\mathrm{Var}(1,a,b,c,d)=4$. Then we shall have $\mathrm{Var}(1,-a,b,-c,d)=0$, which by Corollary 3.2 means that the characteristic equation $f(\lambda)=0$ has no negative root. This contradicts Theorem 2.1. □

Lemma 3.4. *If* $\mathrm{Var}(1,a,b,c,d)=0$*, the two ellipsoids* $\mathcal{A}$ *and* $\mathcal{B}$ *overlap.*

Proof. By Descartes' rule of signs, $\mathrm{Var}(1,a,b,c,d)=0$ implies that the characteristic equation $f(\lambda)=0$ has no positive root. By Theorem 2.1, the two ellipsoids overlap. □

Hence given the characteristic polynomial $f(\lambda)$ of two static ellipsoids, we can immediately tell that the two ellipsoids overlap if $\mathrm{Var}(1,a,b,c,d)=0$. However, the converse of Lemma 3.4 is not true.

Lemma 3.5. *If two ellipsoids* $\mathcal{A}$ *and* $\mathcal{B}$ *overlap, then* $\mathrm{Var}(1,a,b,c,d)=0$ *or* 2; *furthermore if their characteristic equation* $f(\lambda)=0$ *has four real roots, then* $\mathrm{Var}(1,a,b,c,d)=0$.

Proof. Since the ellipsoids $\mathcal{A}$ and $\mathcal{B}$ overlap, $f(\lambda)=0$ has no positive root. By Descartes' rule of signs and Lemma 3.3, $\mathrm{Var}(1,a,b,c,d)=0$ or 2.

Now if $f(\lambda)=0$ has four real roots, since none of these roots can be positive, we have $\mathrm{Var}(1,-a,b,-c,d)=4$ and therefore $\mathrm{Var}(1,a,b,c,d)=0$. □

Using the above lemmas, we have

Lemma 3.6. *Two ellipsoids* $\mathcal{A}$ *and* $\mathcal{B}$ *are separate or externally touching if and only if their characteristic equation* $f(\lambda)=0$ *has four real roots and* $\mathrm{Var}(1,a,b,c,d)=2$.

Proof. "$\Longrightarrow$": By Theorem 2.1, the configuration of separation or external touch implies that the characteristic equation has four real roots. On the other hand, by Proposition 3.1, $\mathrm{Var}(1,a,b,c,d)$ is an even number, hence can be 0, 2 or 4. But by Lemma 3.3 $\mathrm{Var}(1,a,b,c,d)\neq 4$, and by Lemma 3.4 $\mathrm{Var}(1,a,b,c,d)\neq 0$. So $\mathrm{Var}(1,a,b,c,d)=2$.

"$\Longleftarrow$": Since $\mathrm{Var}(1,a,b,c,d)=2$, by Proposition 3.1 $f(\lambda)=0$ has zero or two positive roots. If $f(\lambda)=0$ has no positive root, the two ellipsoids overlap. Now since $f(\lambda)=0$ has four real roots, by Lemma 3.5 we have $\mathrm{Var}(1,a,b,c,d)=0$, a contradiction. Therefore $f(\lambda)=0$ has two positive roots, that is, the two ellipsoids are either separate or externally touching. □

Table 1 summarizes Lemma 3.3 to Lemma 3.6.

Next we are going to show how to determine whether the characteristic equation $f(\lambda)=0$ has four real roots. Consider the characteristic polynomial $f(\lambda)$ and its derivative $f'(\lambda)$, together with their first three signed subresultants $\mathbf{sr}_0(\lambda), \mathbf{sr}_1(\lambda), \mathbf{sr}_2(\lambda)$. Denote the sequence

$$\mathcal{P}=f(\lambda), f'(\lambda), \mathbf{sr}_2(\lambda), \mathbf{sr}_1(\lambda), \mathbf{sr}_0, \tag{1}$$

where by Definition 2.2,

$$f(\lambda)=\lambda^4+a\lambda^3+b\lambda^2+c\lambda+d,$$
$$f'(\lambda)=4\lambda^3+3a\lambda^2+2b\lambda+c,$$
$$\mathbf{sr}_2(\lambda)=\left(-8b+3a^2\right)\lambda^2+(2ab-12c)\lambda+ac-16d,$$
$$\mathbf{sr}_1(\lambda)=\left(-6a^3c+2a^2b^2-12a^2d+28abc-8b^3-36c^2+32bd\right)\lambda-9a^3d+a^2bc+3ac^2+32abd-4b^2c-48cd,$$

$$\mathbf{sr}_0 = -192cd^2a + 256d^3 + 144c^2db + b^2a^2c^2 - 6c^2da^2 + 18c^3ba + 144ba^2d^2 - 4b^3a^2d + 16b^4d - 4c^3a^3 - 128d^2b^2 - 4b^3c^2 - 27a^4d^2 - 80cb^2ad + 18cba^3d - 27c^4. \tag{2}$$

Using the same notation as in Basu et al. (2006), let $\mathrm{MVar}(\mathcal{P}; a)$ denote the modified[1] number of sign variations in a sequence of polynomials $\mathcal{P} = P_0, \ldots, P_n$ evaluated at $a \in \mathbb{R} \cup \{-\infty, +\infty\}$, that is, $\mathrm{MVar}(\mathcal{P}; a) = \mathrm{MVar}(P_0(a), \ldots, P_n(a))$. Also, denote $\mathrm{MVar}(\mathcal{P}; a, b) = \mathrm{MVar}(\mathcal{P}; a) - \mathrm{MVar}(\mathcal{P}; b)$, where $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Proposition 3.7. *(See Basu et al., 2006.) Let f be a polynomial of degree n in $\mathbb{R}[\lambda]$, and let $a < b$ be elements in $\mathbb{R} \cup \{-\infty, +\infty\}$ that are not roots of f. Denote by $\mathcal{P}$ the signed subresultant sequence $\{f, f', \mathbf{sr}_{n-2}(f, f'), \ldots, \mathbf{sr}_0(f, f')\}$. Then the number of real roots of f counting without multiplicities in $\mathbb{R}$ is equal to* $\mathrm{MVar}(\mathcal{P}; a, b)$.

Lemma 3.8. *The characteristic equation $f(\lambda) = 0$ has four real roots if and only if one of the following four cases holds.*

1. $\mathbf{sr}_{22} > 0$, $\mathbf{sr}_{11} > 0$, $\mathbf{sr}_0 > 0$; *this occurs if and only if $f(\lambda) = 0$ has four distinct simple roots.*
2. $\mathbf{sr}_{22} > 0$, $\mathbf{sr}_{11} > 0$, $\mathbf{sr}_0 = 0$; *this occurs if and only if $f(\lambda) = 0$ has one double root and two distinct simple roots.*
3. $\mathbf{sr}_{22} > 0$, $\mathbf{sr}_{11} = 0$, $\mathbf{sr}_0 = 0$; *this occurs if and only if $f(\lambda) = 0$ has two double roots or a simple and a triple root.*
4. $\mathbf{sr}_{22} = 0$, $\mathbf{sr}_{11} = 0$, $\mathbf{sr}_0 = 0$; *this occurs if and only if $f(\lambda) = 0$ has one quadruple root.*

Proof. The above enumeration covers all possible cases under which the characteristic equation $f(\lambda) = 0$ has four real roots. We now establish the corresponding algebraic conditions for these cases.

For case 1, by Proposition 3.7, $\mathrm{MVar}(\mathcal{P}; -\infty, +\infty) = 4$. Note that

$$\mathrm{MVar}(\mathcal{P}; -\infty) = \mathrm{MVar}(+, -, \mathbf{sr}_{22}, -\mathbf{sr}_{11}, \mathbf{sr}_0),$$
$$\mathrm{MVar}(\mathcal{P}; +\infty) = \mathrm{MVar}(+, +, \mathbf{sr}_{22}, \mathbf{sr}_{11}, \mathbf{sr}_0).$$

Hence the only possible choice is that $\mathbf{sr}_{22} > 0$, $\mathbf{sr}_{11} > 0$ and $\mathbf{sr}_0 > 0$.

For case 2, since $f(\lambda) = 0$ has a double root, by Lemma 2.3 we have $\mathbf{sr}_0 = 0$. Again by Proposition 3.7, $\mathrm{MVar}(\mathcal{P}; -\infty, +\infty) = 3$. Note that

$$\mathrm{MVar}(\mathcal{P}; -\infty) = \mathrm{MVar}(+, -, \mathbf{sr}_{22}, -\mathbf{sr}_{11}, 0),$$
$$\mathrm{MVar}(\mathcal{P}; +\infty) = \mathrm{MVar}(+, +, \mathbf{sr}_{22}, \mathbf{sr}_{11}, 0).$$

Hence the only possible choice is that $\mathbf{sr}_{22} > 0, \mathbf{sr}_{11} > 0$.

For case 3, since $f(\lambda) = 0$ has two double roots or a simple and a triple root, the degree of $\gcd(f(\lambda), f'(\lambda))$ is 2. By Lemma 2.3, we have $\mathbf{sr}_0 = 0$ and $\mathbf{sr}_{11} = 0$. By Proposition 3.7, $\mathrm{MVar}(\mathcal{P}; -\infty, +\infty) = 2$. Note that

$$\mathrm{MVar}(\mathcal{P}; -\infty) = \mathrm{MVar}(+, -, \mathbf{sr}_{22}, 0, 0),$$
$$\mathrm{MVar}(\mathcal{P}; +\infty) = \mathrm{MVar}(+, +, \mathbf{sr}_{22}, 0, 0).$$

Hence the only possible choice here is that $\mathbf{sr}_{22} > 0$.

For case 4, by Lemma 2.3, $f(\lambda) = 0$ has one quadruple root if and only if $\mathbf{sr}_{22} = \mathbf{sr}_{11} = \mathbf{sr}_0 = 0$. □

Next we shall provide algebraic conditions for characterizing the configurations of two ellipsoids.

Lemma 3.9. *Let $f(\lambda) = 0$ be the characteristic equation of two ellipsoids $\mathcal{A}$: $X^T AX = 0$ and $\mathcal{B}$: $X^T BX = 0$. If $\mathbf{sr}_{22} \leqslant 0$ then the two ellipsoids overlap.*

Proof. If $\mathbf{sr}_{22} \leqslant 0$, then $\mathrm{MVar}(\mathcal{P}; -\infty, +\infty) \leqslant 2$. By Proposition 3.7, $f(\lambda) = 0$ has at most 2 distinct real roots. As $f(\lambda) = 0$ has at least two negative roots (counted with multiplicity), considering $f(-\infty) > 0$, $f(+\infty) > 0$ and $f(0) = d > 0$, f has either no positive roots or two double roots of opposite signs. For the latter case, we must have both $\mathrm{MVar}(\mathcal{P}; -\infty, +\infty) = 2$ and $\mathbf{sr}_{11} = \mathbf{sr}_0 = 0$. But $\mathbf{sr}_{22} \leqslant 0$ yields $\mathrm{MVar}(\mathcal{P}; -\infty, +\infty) \leqslant 1$, which is a contradiction. Thus $f(\lambda) = 0$ has no positive root and by Lemma 2.1 the two ellipsoids overlap. □

Theorem 3.10. *Let $f(\lambda) = 0$ be the characteristic equation of two ellipsoids $\mathcal{A}$: $X^T AX = 0$ and $\mathcal{B}$: $X^T BX = 0$.*

1. *The two ellipsoids $\mathcal{A}$ and $\mathcal{B}$ are separate if and only if* $\mathrm{Var}(1, a, b, c, d) = 2$ *and*

[1] We first delete those polynomials identical to zero in the sequence $\mathcal{P}$. Then the modified number of sign variations $\mathrm{MVar}(\mathcal{P}, a)$ is similarly defined as the commonly used number of sign variations $\mathrm{Var}(\mathcal{P}, a)$ except that the case $\{+, 0, 0, +\}$ or $\{-, 0, 0, -\}$ (exactly two zeros between the two nonzero number) counts the variation for two but not zero. See page 330 of Basu et al. (2006) for details.

(a) $\mathbf{sr}_{22} > 0$, $\mathbf{sr}_{11} > 0$, $\mathbf{sr}_0 > 0$; *or*
(b) $\mathbf{sr}_{22} > 0$, $\mathbf{sr}_{11} > 0$, $\mathbf{sr}_{10} > 0$, $\mathbf{sr}_0 = 0$.

2. *The two ellipsoids $\mathcal{A}$ and $\mathcal{B}$ touch each other externally if and only if*
   (a) $\mathbf{sr}_{22} > 0$, $\mathbf{sr}_{11} > 0$, $\mathbf{sr}_{10} < 0$, $\mathbf{sr}_0 = 0$; *or*
   (b) $\mathbf{sr}_{22} > 0$, $\mathbf{sr}_{20} < 0$, $\mathbf{sr}_{11} = 0$, $\mathbf{sr}_0 = 0$.

In the other cases, the two ellipsoids overlap.

Proof.

1. "$\Longrightarrow$": Since $\mathcal{A}$ and $\mathcal{B}$ are separate, by Lemma 3.6, $\mathrm{Var}(1, a, b, c, d) = 2$ and $f(\lambda) = 0$ has four real roots, two of which are distinct positive reals and the other two are negative. This leads to two subcases:
   (a) $f(\lambda) = 0$ has two distinct negative roots. By Lemma 3.8, we have $\mathbf{sr}_{22} > 0, \mathbf{sr}_{11} > 0, \mathbf{sr}_0 > 0$;
   (b) $f(\lambda) = 0$ has one negative double root. By Lemma 3.8, $\mathbf{sr}_{22} > 0, \mathbf{sr}_{11} > 0, \mathbf{sr}_0 = 0$. By Lemma 2.3, $\gcd(f(\lambda), f'(\lambda)) = \mathbf{sr}_1 = \mathbf{sr}_{11}\lambda + \mathbf{sr}_{10} = \mathbf{sr}_{11}(\lambda - \lambda_0)$, where $\lambda_0 = -\mathbf{sr}_{10}/\mathbf{sr}_{11}$ is the negative double root of $f(\lambda) = 0$. Hence $\mathbf{sr}_{11} > 0$ yields $\mathbf{sr}_{10} > 0$.

   "$\Longleftarrow$": By Lemma 3.8, both (a) and (b) indicate that $f(\lambda) = 0$ has four real roots. Since $\mathrm{Var}(1, a, b, c, d) = 2$, by Lemma 3.6 the two ellipsoids are separate or externally touch.
   (a) Since $\mathbf{sr}_0 \neq 0$, $f(\lambda) = 0$ has no multiple root. Hence by Theorem 2.1 the two ellipsoids are separate.
   (b) Since $\mathbf{sr}_0 = 0$ and $\mathbf{sr}_{11} \neq 0$, $f(\lambda) = 0$ has one double root and two simple roots. The double root λ_0 is the root of $\mathbf{sr}_1 = \mathbf{sr}_{11}\lambda + \mathbf{sr}_{10}$, and hence negative because $\mathbf{sr}_{11} > 0, \mathbf{sr}_{10} > 0$. Therefore by Theorem 2.1 the two ellipsoids cannot be externally touching, and are therefore separate.
2. "$\Longrightarrow$": Since $\mathcal{A}$ and $\mathcal{B}$ are externally touching, by Lemma 3.6 $f(\lambda) = 0$ has four real roots, two of which are a positive double root and the other two are negative. This also leads to two subcases:
   (a) $f(\lambda) = 0$ has two distinct negative roots, which by Lemma 3.8 yields $\mathbf{sr}_{22} > 0, \mathbf{sr}_{11} > 0, \mathbf{sr}_0 = 0$. By Lemma 2.3, $\mathbf{sr}_1 = \mathbf{sr}_{11}\lambda + \mathbf{sr}_{10} = \gcd(f(\lambda), f'(\lambda))$ and therefore $\lambda_0 = -\mathbf{sr}_{10}/\mathbf{sr}_{11}$ is the positive double root of $f(\lambda) = 0$. Now since $\mathbf{sr}_{11} > 0$, we have $\mathbf{sr}_{10} < 0$.
   (b) $f(\lambda) = 0$ has one negative double root. By Lemma 3.8 $\mathbf{sr}_{22} > 0, \mathbf{sr}_{11} = 0, \mathbf{sr}_0 = 0$. By Lemma 2.3 $\mathbf{sr}_2 = \mathbf{sr}_{22}\lambda^2 + \mathbf{sr}_{21}\lambda + \mathbf{sr}_{20} = \gcd(f(\lambda), f'(\lambda)) = \mathbf{sr}_{22}(\lambda - \lambda_0)(\lambda - \lambda_1)$, where λ_0 is the positive double root and λ_1 is the negative double root. Hence $\mathbf{sr}_{20}/\mathbf{sr}_{22} = \lambda_0\lambda_1 < 0$. Since $\mathbf{sr}_{22} > 0$, we have $\mathbf{sr}_{20} < 0$.

   "$\Longleftarrow$":
   (a) By Lemma 3.8, $f(\lambda) = 0$ has one double root and two simple roots. By Lemma 2.3, $\mathbf{sr}_1 = \mathbf{sr}_{11}\lambda + \mathbf{sr}_{10} = \gcd(f(\lambda), f'(\lambda))$ and therefore $\lambda_0 = -\mathbf{sr}_{10}/\mathbf{sr}_{11} > 0$ is the double root of $f(\lambda) = 0$. By Lemma 2.1 the two ellipsoids are externally touching.
   (b) By Lemma 3.8 there are two subcases:
      (i) $f(\lambda) = 0$ has two double roots. By Lemma 2.3 $\mathbf{sr}_2 = \mathbf{sr}_{22}\lambda^2 + \mathbf{sr}_{21}\lambda + \mathbf{sr}_{20} = \gcd(f(\lambda), f'(\lambda)) = \mathbf{sr}_{22}(\lambda - \lambda_0)(\lambda - \lambda_1)$, where λ_0 and λ_1 are the two double roots. Since $\lambda_0\lambda_1 = \mathbf{sr}_{20}/\mathbf{sr}_{22} < 0$, one of the double root should be positive. By Theorem 2.1 the two ellipsoids are in external touch.
      (ii) $f(\lambda) = 0$ has one simple root and one triple root λ_0. Then $\mathbf{sr}_{20}/\mathbf{sr}_{22} = \lambda_0^2 > 0$, which contradicts the fact that $\mathbf{sr}_{22} > 0$ and $\mathbf{sr}_{20} < 0$. Hence this subcase never happens. □

Remark 3.1. According to Theorem 3.10, the subresultant coefficient $\mathbf{sr}_{10}$ is crucial for distinguishing the two configurations of separation and external touching. Note that Gonzalez-Vega and Mainar (2008) uses principal subresultant sequences (i.e., $\mathbf{sr}_{22}, \mathbf{sr}_{11}, \mathbf{sr}_0$) to derive the explicit formulae which do not distinguish separation from external touching of two static ellipsoids. We achieves this distinction by considering the sign of $\mathbf{sr}_{10}$.

Table 2 summarizes Theorem 3.10 on characterizing the configuration, i.e., separation, external touching or overlapping, of two static ellipsoids.

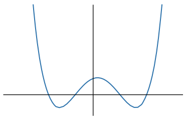
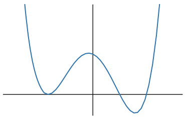
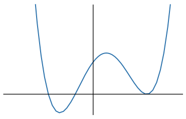
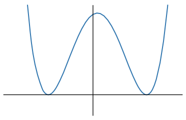
4. Computation costs

Our method for determining the configuration of two ellipsoids involves only the evaluation of the explicit formulae $\mathbf{sr}_{22}, \mathbf{sr}_{20}, \mathbf{sr}_{11}, \mathbf{sr}_{10}, \mathbf{sr}_0$ and $\mathrm{Var}(1, a, b, c, d)$. Here we adopt the optimized evaluation of these five polynomials provided by Emiris and Tsigaridas (2008) obtained from the Bezoutian matrix. Let

$$\bar{b} = -\frac{a}{4}, \qquad \bar{c} = \frac{b}{6}, \qquad \bar{d} = -\frac{c}{4}, \qquad \bar{e} = d,$$

and let

Table 2
Algebraic conditions for characterizing the configuration of two ellipsoids. The case numbers correspond to that of Lemma 3.8.

Cases		$\mathrm{Var}(1, a, b, c, d)$	$\mathrm{sr}_2(\lambda)$		$\mathrm{sr}_1(\lambda)$			Ellipsoids configurations
			sr_{22}	sr_{20}	sr_{11}	sr_{10}	sr_0	
(1)		2	> 0		> 0		> 0	separate
(2)		2	> 0		> 0	> 0	$= 0$	separate
			> 0		> 0	< 0	$= 0$	externally touch
(3)			> 0	< 0	$= 0$		$= 0$	externally touch

The remaining cases correspond to overlapping ellipsoids.

$$\begin{aligned}
&\Delta_2 = \bar{b}^2 - \bar{c}, &&W_1 = \bar{d} - \bar{b}\bar{c}, &&T = -9W_1^2 + 27\Delta_2\Delta_3 - 3W_3\Delta_2,\\
&\Delta_3 = \bar{c}^2 - \bar{b}\bar{d}, &&W_2 = \bar{b}\bar{e} - \bar{c}\bar{d}, &&A = W_3 + 3\Delta_3,\\
& &&W_3 = \bar{e} - \bar{b}\bar{d}, &&B = -\bar{d}W_1 - \bar{e}\Delta_2 - \bar{c}\Delta_3,\\
& && &&T_2 = AW_1 - 3\bar{b}B,\\
& && &&\Delta_1 = A^3 - 27B^2.
\end{aligned}$$

The two explicit formulae are then given by

$$\mathbf{sr}_{22} := \Delta_2, \qquad \mathbf{sr}_{20} := -W_3, \qquad \mathbf{sr}_{11} := T, \qquad \mathbf{sr}_{10} := T_2, \qquad \mathbf{sr}_0 := \Delta_1$$

up to a positive constant multiple. The above expressions take 28 multiplications and 12 additions.

5. Application: continuous collision detection for two moving ellipsoids

5.1. Algorithm

Let $M(t)$ be a 4×4 matrix, whose entries are arbitrary smooth functions in t, that represents an arbitrary continuous deformation and motion in $\mathbb{R}^3$. By applying such deformation and motion to an ellipsoid $\mathcal{A}$: $X^T AX = 0$, we obtain a moving ellipsoid $\mathcal{A}(t)$: $X^T M^{-1}(t)^T AM^{-1}(t)X = X^T A(t)X = 0$. Next we extend our method to solving continuous collision detection for two moving ellipsoids $\mathcal{A}(t)$: $X^T A(t)X = 0$ and $\mathcal{B}(t)$: $X^T B(t)X = 0$, whose center positions vary and shapes deform with respect to a continuously varied parameter $t \in [0, 1]$. The two moving ellipsoids $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are said to be *collision-free* if $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are separate for all $t \in [0, 1]$; otherwise $\mathcal{A}(t)$ and $\mathcal{B}(t)$ collide.

In this continuous setting, the characteristic polynomial associated with $\mathcal{A}(t)$ and $\mathcal{B}(t)$ becomes a bivariate polynomial both in parameter λ and parameter t which is given by $f(\lambda; t) = \det(\lambda A(t) + B(t))$. Clearly, $f(\lambda; t)$ is of degree 4 in λ with coefficients as functions of t. We divide $f(\lambda; t)$ by its leading coefficient in λ and get

$$f(\lambda; t) = \lambda^4 + a(t)\lambda^3 + b(t)\lambda^2 + c(t)\lambda + d(t), \tag{3}$$

where $a(t), b(t), c(t), d(t)$ are all functions in t. We shall study the signed subresultants $\mathbf{sr}_0(t)$ and $\mathbf{sr}_i(\lambda; t)$, $i = 1, 2$, of $f(\lambda; t)$ and $f_\lambda(\lambda; t)$, and denote the coefficient of the degree j term in λ of the polynomial $\mathbf{sr}_i(\lambda; t)$ by $\mathbf{sr}_{ij}(t)$, $i = 1, 2$, $j = 0, \ldots, \deg_\lambda(\mathbf{sr}_i)$.

Theorem 5.1. *Let $0 \leqslant t_1 < t_2 < \cdots < t_m \leqslant 1$, $m \geqslant 0$, be all the distinct contact time instants in $[0, 1]$ at which two given moving ellipsoids are in external touch. Let $t_0 = 0$, $t_{m+1} = 1$ and let δ_i be an arbitrary number in interval (t_i, t_{i+1}), $i = 0, \ldots, m$. Then the configuration of the two moving ellipsoids (i.e., whether they are separate or overlapping) does not change during time interval (t_i, t_{i+1}), and therefore can be decided by their configuration at the time instant $t = \delta_i$, $i = 0, \ldots, m$.*

Proof. Since the center positions and the shapes of the two moving ellipsoids vary continuously, if the configuration of two ellipsoids changes from separation to overlapping, or vice versa, there must be a contact time instant t^* at which the ellipsoids are in external touch. Therefore, within a time interval (t_i, t_{i+1}) that does not contain any other contact time instants, the configuration of the ellipsoids remains the same. Hence we need only check the status of the two moving ellipsoids at a time instant $\delta_i \in (t_i, t_{i+1})$ to decide their configuration during the entire time interval (t_i, t_{i+1}). $\square$

We can see from the above theorem that the primary task for continuous collision detection for two moving ellipsoids is to determine the so-called *contact time instants* at which the ellipsoids are touching externally. The configuration of the ellipsoids between two contact time instants can then be easily identified using the algebraic conditions as established in Theorem 3.10.

Next we explain how to find these contact time instants.

Theorem 5.2. *Suppose that* $\mathbf{sr}_0(t) \not\equiv 0$. *If two ellipsoids* $\mathcal{A}(t)$ *and* $\mathcal{B}(t)$ *touch externally at* t_0, *we have* $\mathbf{sr}_0(t_0) = 0$.

Proof. This follows immediately from Theorem 3.10. $\square$

Note 1. The special situation $\mathbf{sr}_0(t) \equiv 0$ happens when the characteristic equation $f(\lambda; t) = 0$ always has a double root $\lambda(t)$ for any time instant $t \in [0, 1]$, i.e., $f(\lambda; t) = (\lambda - \lambda^*(t))^2 \tilde{f}(\lambda; t)$. Geometrically this occurs when two moving ellipsoids, at every time instant t, touch internally or externally, or have a reducible intersection in $\mathbb{C}^3$ (see Tu et al., 2009 and Example 5.1).

Lemma 5.3. *Given two moving ellipsoids* $\mathcal{A}(t)$ *and* $\mathcal{B}(t)$, $t \in [0, 1]$, *we have either one of the following cases*:

1. $\mathcal{A}(t)$ *and* $\mathcal{B}(t)$ *are in external touch for all* $t \in [0, 1]$;
2. $\mathcal{A}(t)$ *and* $\mathcal{B}(t)$ *are in external touch at some discrete time instants* $t_i \in [0, 1]$;
3. $\mathcal{A}(t)$ *and* $\mathcal{B}(t)$ *never touch during* $[0, 1]$.

Proof. Suppose that $\mathcal{A}(t)$ and $\mathcal{B}(t)$ touch externally at all t within a time interval $I_0 := [t_0, t_1] \subset [0, 1]$ with $t_0 < t_1$. Suppose moreover that $[t_0, t_1]$ is maximal for this property. Then according to Table 2, $\mathbf{sr}_0(t)$ vanishes on I_0, thus $\mathbf{sr}_0(t) \equiv 0$ and $f(\lambda, t) = 0$ has 4 real roots $\alpha_1(t) \leqslant \alpha_2(t) \leqslant \alpha_3(t) \leqslant \alpha_4(t)$ for $t \in I_0$. As $f(0, t) \neq 0$ and $f(\lambda, t) = 0$ has a double positive root for $t \in I_0$, we have

$$\alpha_1(t) \leqslant \alpha_2(t) < 0 < \alpha_3(t) = \alpha_4(t)$$

on $t \in I_0$. Suppose that $[t_0, t_1] \neq [0, 1]$, for instance that $t_0 \in (0, 1)$ (the same arguments will apply if $t_1 \in (0, 1)$). The root functions $\alpha_i(t)$ $(i = 1, \ldots, 4)$ admit a Puiseux expansion in a neighborhood of $t = t_0$ (see e.g. Abhyankar, 1990, Chaps. 12–14). As $\alpha_3(t) = \alpha_4(t)$ on $[t_0, t_1]$, the Puiseux expansion of $\alpha_3(t)$ and $\alpha_4(t)$ at t_0 are identical. Thus they coincide in a neighborhood of t_0 and the two ellipsoids are touching on an interval strictly containing I_0. This is a contradiction. We deduce that $I_0 := [0, 1]$ and that the moving ellipsoids can either be in external touch for all $t \in [0, 1]$ or be in external touch only at some (if there is any) discrete contact time instants. $\square$

Theorem 5.4. *Suppose that* $\mathbf{sr}_0(t) \equiv 0$. *If the moving ellipsoids* $\mathcal{A}(t)$ *and* $\mathcal{B}(t)$ *are in external touch only at some discrete contact time instants* $t_i \in [0, 1]$, *where* $i = 1, \ldots, n$ *and* $n \geqslant 0$, *then at each* t_i, *we have* $\mathbf{sr}_{11}(t_i) = 0$.

Proof. Since $\mathbf{sr}_0(t) \equiv 0$, we have $f(\lambda; t) = (\lambda - \lambda^*(t))^2 \tilde{f}(\lambda; t)$ (Note 1). Since $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are not in external touch for all t, $\lambda^*(t)$ cannot always be a positive double root. Also, since $\lambda^*(t) \neq 0$ for any t, by the continuity of the root function $\lambda^*(t)$, we must have $\lambda^*(t) < 0$ for all t. Now, consider at a contact time instant t_i, $f(\lambda; t) = 0$ has an additional positive double root and therefore $f(\lambda; t_i) = 0$ has two double roots. By Lemma 3.8, we therefore have $\mathbf{sr}_{11}(t_i) = 0$. $\square$

Note 2. Consider two moving spheres that are in external touch at only some discrete contact time instants. Since they always have a reducible intersection in $\mathbb{C}^3$, no matter whether they are separate or not, their characteristic equation always contains a double root (which is negative). Hence, $\mathbf{sr}_0(t) \equiv 0$. Furthermore, $\mathbf{sr}_{11}(t_i) = 0$ if the two spheres are in external touch at t_i.

Theorem 5.5. *Suppose that* $\mathbf{sr}_0(t) \equiv 0$ *and* $\mathbf{sr}_{11}(t) \not\equiv 0$. *Let* $t_1, t_2, \ldots, t_n$, $n \geqslant 0$, *be all the distinct real roots of* $\mathbf{sr}_{11}(t) = 0$ *in* $[0, 1]$. *Let* δ *be an arbitrary number in* $[0, 1] \setminus \{t_1, t_2, \ldots, t_n\}$. *If the ellipsoids* $\mathcal{A}(\delta)$ *and* $\mathcal{B}(\delta)$ *touch externally, then* $\mathcal{A}(t)$ *and* $\mathcal{B}(t)$ *are in external touch throughout* $[0, 1]$.

Proof. This follows from Lemma 5.3 and Theorem 5.4. $\square$

Algorithm 1: Collision detection of two moving ellipsoids.

```
Input  : The characteristic polynomial f(λ; t) = λ^4 + a(t)λ^3 + b(t)λ^2 + c(t)λ + d(t) of two moving ellipsoids A(t) and B(t).
Output: Three sets S, I and T containing time instants or intervals in which A(t) and B(t) are separate, overlap and touching externally,
        respectively.
begin
    Z ← ∅, Z̃ ← ∅, S ← ∅, I ← ∅, T ← ∅;
    if sr_0(t) ≡ 0 then
        if sr_11(t) ≡ 0 then                                                  // Theorem 5.6
            if A(0) and B(0) are separate then S ← [0, 1];
            else if A(0) and B(0) overlap then I ← [0, 1];
            else if A(0) and B(0) are touching externally then T ← [0, 1];
            return;
        else
            Z ← {t | sr_11(t) = 0};                                            // Theorem 5.5
            Select δ ∈ [0, 1] \ Z;
            if A(δ) and B(δ) touch externally then                 // Conditions of Theorem 3.10(2)
                T ← [0, 1];
                return;
    else
        Z ← {t | sr_0(t) = 0}
    foreach t_i ∈ Z do
        if A(t_i) and B(t_i) touch externally then T ← T ∪ {t_i};  // Conditions of Theorem 3.10(2)
    Z̃ ← T ∪ {0, 1};
    Let Z̃ = {t_1, ..., t_n} where 0 = t_1 < ··· < t_n = 1;
    foreach i = 1, ..., n − 1, select δ_i ∈ (t_i, t_{i+1}) do                 // Theorem 5.1
        if A(δ_i) and B(δ_i) are separate then                    // Conditions of Theorem 3.10(1)
            S ← (t_i, t_{i+1});
        else
            I ← (t_i, t_{i+1})
end
```

Theorem 5.6. *Suppose that* $\mathbf{sr}_0(t) \equiv 0$ *and* $\mathbf{sr}_{11}(t) \equiv 0$*. Then either the two ellipsoids* $\mathcal{A}(t)$ *and* $\mathcal{B}(t)$ *touch externally throughout* $[0, 1]$*, or they do not have any external touch at all in* $[0, 1]$*.*

Proof. Since $\mathbf{sr}_0(t) \equiv 0$ and $\mathbf{sr}_{11}(t) \equiv 0$, by Lemma 3.8, $f(\lambda; t) = 0$ always have two double roots or one quadruple root in λ. Due to the continuity of the root functions of a polynomial (see e.g., Bhatia, 1997) and the fact that $\lambda = 0$, $\lambda = \pm\infty$ are not roots of $f(\lambda; t) = 0$, the signs of the roots remain the same in $[0, 1]$. Now, if $f(\lambda; t_i) = 0$ has a positive double root $\lambda_0(t_i)$ for some $t_i \in [0, 1]$, then $\lambda_0(t)$ remains positive throughout $[0, 1]$, which means that $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are in external touch throughout $[0, 1]$. On the other hand, if $f(\lambda; t) = 0$ does not have a positive double root for any $t \in [0, 1]$, then $\mathcal{A}(t)$ and $\mathcal{B}(t)$ do not touch externally at all in the interval. □

Note 3. Theorem 5.6 implies that when $\mathbf{sr}_0(t) \equiv 0$ and $\mathbf{sr}_{11}(t) \equiv 0$, the configuration of the ellipsoids throughout the time span can be determined by their configuration at any particular time instant, e.g., at $t = 0$.

We now summarize the above analysis in Algorithm 1 for continuous collision detection for two moving ellipsoids. Using Theorems 5.2, 5.4, 5.5 and 5.6, we obtain a set Z of time instants which captures all the contact time instants of two moving ellipsoids by solving for the roots of some functions under different conditions. The set Z may also contain other time instants not corresponding to any contact, which can be eliminated easily by checking with the algebraic conditions given by Theorem 3.10. Again by Theorem 3.10, the configuration of the ellipsoids at each interval defined by two consecutive contact time instants can then be determined.

5.2. Examples

Example 5.1. Let $\mathcal{A}(t)$ and $\mathcal{B}(t)$ be two moving ellipsoids defined by

$$\frac{(x + 12t - 11)^2}{4} + y^2 + z^2 = 1 \quad \text{and} \quad \frac{(x-3)^2}{4} + (y - 4t + 2)^2 + (z - 4t + 4)^2 = 1,$$

respectively, where $t \in [0, 1]$. The characteristic polynomial associated with $\mathcal{A}(t)$ and $\mathcal{B}(t)$ is

$$\begin{aligned} f(\lambda; t) &= \lambda^4 + \left(-68t^2 + 96t - 32\right)\lambda^3 + \left(-136t^2 + 192t - 66\right)\lambda^2 + \left(-68t^2 + 96t - 32\right)\lambda + 1 \\ &= -(\lambda + 1)^2\left(-\lambda^2 + \left(68t^2 - 96t + 34\right)\lambda - 1\right) \end{aligned}$$

up to a constant multiple. Since $\mathbf{sr}_0(t) \equiv 0$, we shall next compute

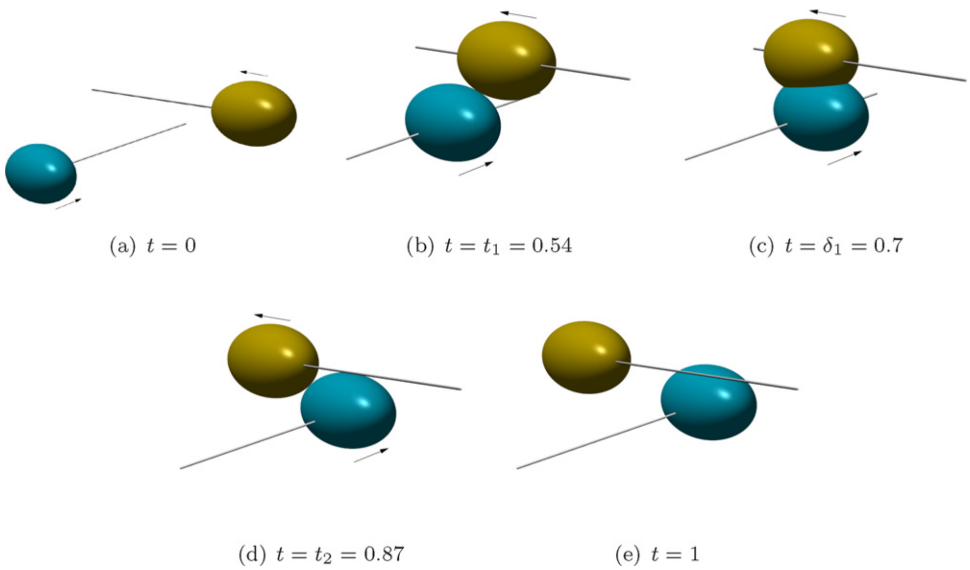


Fig. 1. Configurations of two moving ellipsoids under rational motion in Example 5.1. The ellipsoids touch externally at $t = t_1$ and $t = t_2$.

$$\mathbf{sr}_{11}(t) = 42\,762\,752t^8 - 241\,483\,776t^7 + 599\,418\,368t^6 - 854\,175\,744t^5 + 764\,204\,544t^4 - 439\,492\,608t^3 + 158\,630\,400t^2 - 32\,845\,824t + 2\,985\,984.$$

Solving the real roots for $\mathbf{sr}_{11}(t)$ in $[0, 1]$, we obtain $t_1 = 0.5395042868, t_2 = 0.8722604191$ which are both confirmed to be the contact time instants of the ellipsoids by checking against the conditions in Theorem 3.10. Selecting

$$\delta_0 = 0.2 \in (0, t_1), \qquad \delta_1 = 0.7 \in (t_1, t_2), \qquad \delta_2 = 0.95 \in (t_2, 1),$$

and by checking the collision states at $t = \delta_0, \delta_1, \delta_2$, we conclude that the two ellipsoids are separate during time interval $[0, t_1)$, overlap during (t_1, t_2), and are separate again during $(t_2, 1]$ (Fig. 1).

The following example shows that our approach not only works for rational motions but also allows arbitrary functional motions, e.g., helical motions, of two moving ellipsoids.

Example 5.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two static ellipsoids defined by

$$x^2 + \frac{y^2}{4} + z^2 = 1 \quad \text{and} \quad x^2 + y^2 + \frac{(z-5)^2}{9} = 1,$$

respectively. Let $\mathcal{A}(t)$ be a moving ellipsoid defined by applying to $\mathcal{A}$ a rotation about the axis $(1, 0, 0)^T$ by an angle $10t$ and then a translation along the helical curve $P(t) = (\cos 10t, \sin 10t, 10t)$, where $t \in [0, 1]$. The coefficients of the characteristic polynomial associated with the moving ellipsoid $\mathcal{A}(t)$ and the static ellipsoid $\mathcal{B}$ are functions in t that contains trigonometric terms; the expressions are long and hence are omitted here.

We solve the transcendental function $\mathbf{sr}_0(t)$ by the simple bracketing and bisection method. Other methods, such as the secant method, can also be used for root finding (see Press et al., 2007 for more details). The real roots of $\mathbf{sr}_0(t)$ in $[0, 1]$ are found to be $t_1 = 0.0749830692$ and $t_2 = 0.8913371204$, which are both confirmed to be the contact time instants of the ellipsoids. We check the collision states at

$$\delta_0 = 0.02 \in (0, t_1), \qquad \delta_1 = 0.4 \in (t_1, t_2), \qquad \delta_2 = 0.95 \in (t_2, 1),$$

and conclude that the two ellipsoids are separate in $[0, t_1)$, overlap in (t_1, t_2), and are separate again in $(t_2, 1]$ (Fig. 2).

5.3. Comparison

In this section, we compare our method with Choi et al. (2009) on continuous collision detection for two ellipsoids. Since the method in Choi et al. (2009) only deals with ellipsoids under rational motions, the examples we use here are also confined to rational motions. Both algorithms are implemented in C++ and the tests are run on a workstation with an Intel Xeon 3.33-GHz CPU. Double precision floating-point arithmetic is used for all computations in the comparison. Polynomials

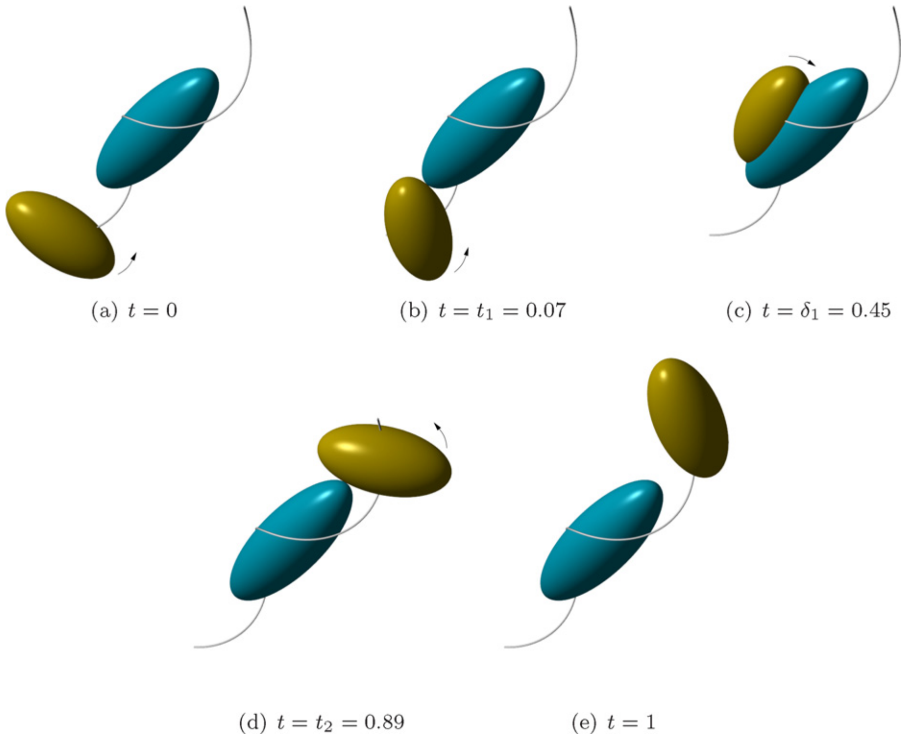


Fig. 2. Configurations of two ellipsoids, one static and the other moves under non-rational helical motion, in Example 5.2. The ellipsoids touch externally at $t = t_1$ and $t = t_2$.

are represented in the Bernstein form in order to improve the robustness and accuracy of the computations. Root solving of polynomials are then done by subdivision using the de Casteljau algorithm. The algorithms are applied to three pairs of moving ellipsoids under different motion types to detect their collision states over a specific time span and the performance of the algorithms are listed in Table 3. Each test is run for 1000 times and the average running time is taken.

The degree 2 rational rigid motion includes the simple yet commonly used motion in which an object assumes a degree 2 rotation plus a linear translation. Under this kind of rigid motion, the two methods have comparable performances and can both complete CCD in about 0.1 ms. The next pair of ellipsoids we study are under degree 2 rational affine motion, that is, one with deformation. The degree of $f(\lambda; t)$ in t is 36 in this particular example. The method in Choi et al. (2009) needs much longer (about 2.5 ms) to compute CCD, because their method deals with a bivariate function $f(\lambda; t)$ and basically needs to find a pathway $\lambda(t)$ such that $f(\lambda; t) > 0$ for all t to declare that the ellipsoids are always separate. The time taken therefore depends not only on the degree of the motion but also on the topology of the zero set of $f(\lambda; t)$. In this example, the two moving ellipsoids are in close proximity from time to time but remain separate within the entire time span. The approach in Choi et al. (2009) therefore takes longer to find the pathway $\lambda(t)$. On the other hand, our method does not depend on the complexity of $f(\lambda; t)$ and can solve the CCD in 0.7 ms.

In the last example, the moving ellipsoids are under rational motions of degree 4 with large deformations (Example 2 of Choi et al., 2009). The degree of $f(\lambda; t)$ in t is 48. The Choi et al. (2009) method takes about 1 ms while ours takes about 13 ms to complete CCD. The slower performance of our method is due to the high degree in the subresultant expressions. The degree of $\mathrm{sr}_0(t)$ is 288; its composition and root finding are therefore time consuming. When only the first contact time instant of two moving ellipsoids is required (which is a common output for CCD), our method does not need to solve for all roots of the subresultants and can complete CCD in 0.4 ms.

We remark here that the above examples serve to demonstrate the efficiency of our method when time performance is of major concern. Both Choi et al. (2009) and ours are exact continuous collision detection methods in the sense that no discretization of the time domain is needed. However, as we mentioned in Section 1, Choi et al. (2009) solve a bivariate characteristic equation using numerical computations. We therefore use a float-point implementation of our method for comparison with Choi et al. (2009). Note that, when high numerical accuracy is desired, our method has the advantage that exact arithmetic can be used to achieve any required accuracy. We also listed in Table 4 the corresponding time costs for the same collision detection examples under degree 2 rigid motion and degree 2 affine motion using symbolic computation.

Table 3
Run-time performance of our method against Choi et al. (2009) to solve CCD of two moving ellipsoids under rational motions. The timing under the column "Solve $f(\lambda;t)$" is the time taken for determining the collision states over a given time span. The total time for CCD is the sum of the time taken for setting up and solving $f(\lambda;t)$. All timings are averaged over 1000 runs.

Motion type	$\deg_t(f(\lambda;t))$	$\deg(\mathrm{sr}_0(t))$	Time (ms)				
			Set up $f(\lambda;t)$	Solve $f(\lambda;t)$		Total for CCD	
				Choi et al. (2009)	Our	Choi et al. (2009)	Our
Degree 2 rigid	4	24	0.014	0.094	0.096	0.108	0.11
Degree 2 affine	36	216	0.12	2.411	0.571	2.531	0.691
Degree 4 affine	48	288	0.202	0.965	12.707	1.167	12.909

Table 4
Run-time performance of our method to symbolically solve CCD of two moving ellipsoids under rational motions. All timings are averaged over 1000 runs.

Motion type	$\deg_t(f(\lambda;t))$	$\deg(\mathrm{sr}_0(t))$	Time (s)		
			Set up $f(\lambda;t)$	Solve $f(\lambda;t)$	Total for CCD
Degree 2 rigid	4	24	0.002169	0.016	0.018169
Degree 2 affine	36	216	0.002683	1.326	1.328683

6. Conclusions

We use five explicit formulae to decide the geometric configurations, that are, separate, externally touching or overlapping of two ellipsoids. Our derivation is based on the algebraic conditions provided in Wang et al. (2001), which shows the correspondence between the root patterns of the characteristic polynomial and the configurations of two ellipsoids. The explicit formulae are composed of the coefficients of the signed subresultant sequence of the characteristic polynomial and its first derivative. These algebraic formulae can be applied naturally to continuous collision detection for two moving ellipsoids whose relative positions as well as shapes vary along time governed by arbitrary continuous functions. In future work, we expect to apply simple techniques and develop explicit formulae to determine the relative geometric configuration of two general quadrics.

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