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An algebraic condition for the separation of two ellipsoids

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Abstract

Given two ellipsoids, we show that their characteristic equation has at least two negative roots and that the ellipsoids are separated by a plane if and only if their characteristic equation has two distinct positive roots. Furthermore, the ellipsoids touch each other externally if and only if the characteristic equation has a positive double root. An advantage of this characterization is that only the signs of the roots matter. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

We present an algebraic condition for the separation of ellipsoids in three-dimensional Euclidean space, E^3. Two ellipsoids are said to be separated if they are on the opposite sides of a plane and do not intersect the plane in E^3; and to overlap if they have common interior points. Two ellipsoids that touch each other externally are not regarded as overlapping or as separated.

Ellipsoids have a small number of geometric parameters and are excellent for approximating a wide class of convex objects in simulations of physical systems. Detecting the collision or overlap of two ellipsoids is thus an important problem with applications in computer graphics, computer animation, virtual reality, robotics, CAD/CAM, computational physics, and geomechanics. However, the use of ellipsoids is hindered by the lack of efficient methods for detecting separation or overlap.

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The intersection curves between two quadric surfaces in three-dimensional projective complex space can be classified using Segre's characteristics, which are defined by the elementary divisors of the associated quadratic forms (Bromwich, 1906; Sommerville, 1947). For all degenerate intersection curves, this classification essentially takes place in three-dimensional complex space, so it cannot be directly applied to the present problem of detecting separation in real space. In some sense, our work is an extension of the classical results in (Bromwich, 1906), since we consider the classification of a pair of ellipsoids in real affine space.

Conventional methods (Farouki et al., 1989; Levin, 1979; Wilf and Manor, 1993) for finding the intersection of two quadrics could also be used to detect whether two ellipsoids overlap; if there are no real intersection points between them, then the ellipsoids are either separated or one is contained in the other. However, these methods are designed to compute the structure and parameterization of the intersection curve, rather than the gross relationship between the ellipsoids, and are more complicated than the algebraic condition that will be given in this paper.

The overlap of two ellipsoids has also been studied in computational physics for molecule simulation (Perram and Wertheim, 1985; Perram et al., 1996), and in geomechanics for modeling ellipsoidal particles (Lin and Ng, 1995); the methods proposed for these applications are essentially numerical. In contrast, our algebraic condition leads to simple, efficient, and exact algorithms.

Given two ellipsoids $\mathcal{A}$: $X^{\mathrm{T}}AX = 0$ and $\mathcal{B}$: $X^{\mathrm{T}}BX = 0$ in E^3, where $X = (x, y, z, w)^{\mathrm{T}}$ are the homogeneous coordinates, their *characteristic polynomial* is defined as

$$f(\lambda) = \det(\lambda A + B),$$

and $f(\lambda) = 0$ is called the *characteristic equation*. Here we assume that the interiors of $\mathcal{A}$ and $\mathcal{B}$ are defined by $X^{\mathrm{T}}AX < 0$ and $X^{\mathrm{T}}BX < 0$, respectively. We shall show that:

(i) The characteristic equation $f(\lambda) = 0$ always has at least two negative roots.

(ii) The two ellipsoids are separated by a plane if and only if $f(\lambda) = 0$ has two distinct positive roots.

(iii) The two ellipsoids touch each other externally if and only if $f(\lambda) = 0$ has a positive double root.

Note that only the signs are important—we do not need to solve for the exact roots. As soon as two distinct positive roots are detected (e.g., using Sturm sequences (Dickson, 1914)), one may conclude that the two ellipsoids are separated.

The remainder of this paper is organized as follows. In Section 2, we briefly review some preliminaries on ellipsoids. In Section 3, we introduce an algebraic condition for the separation of two ellipsoids, and show that it is a necessary and sufficient condition. Some examples are given in Section 5. Finally, in Section 6, we conclude with some ideas for future research.

2. Preliminaries

Given two ellipsoids, by applying an affine transformation as necessary, we may assume that one ellipsoid is given in the standard form

$$\mathcal{A}:\ X^{\mathrm{T}}AX = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \tag{1}$$

with $0 < a \leqslant b \leqslant c$, and the other is given as a sphere

$$\mathcal{B}:\ X^{\mathrm{T}}BX = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 - r^2 = 0, \tag{2}$$

where $X = (x, y, z, 1)^{\mathrm{T}}$, and

$$A = \begin{pmatrix} 1/a^2 & & & \\ & 1/b^2 & & \\ & & 1/c^2 & \\ & & & -1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & & & -x_c \\ & 1 & & -y_c \\ & & 1 & -z_c \\ -x_c & -y_c & -z_c & -r^2 + x_c^2 + y_c^2 + z_c^2 \end{pmatrix}.$$

An affine transformation of two ellipsoids $\mathcal{A}$ and $\mathcal{B}$ changes their characteristic equation; however, the roots remain the same. Let $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ be the result of applying an affine transformation T to $\mathcal{A}$ and $\mathcal{B}$. Then the corresponding matrices are $\widetilde{A} = (T^{-1})^{\mathrm{T}}AT^{-1}$ and $\widetilde{B} = (T^{-1})^{\mathrm{T}}BT^{-1}$, and the characteristic equation becomes $\det(\lambda\widetilde{A} + \widetilde{B}) = \det^{-2}(T)\det(\lambda A + B) = 0$, which has the same roots as $f(\lambda) = \det(\lambda A + B) = 0$. Clearly, the geometric relationship between the two ellipsoids does not change under the affine transformation T. Thus it is sufficient to consider the simple case of an ellipsoid $\mathcal{A}$ in standard form (1), and a sphere $\mathcal{B}$ in form (2).

The characteristic polynomial of $\mathcal{A}$ and $\mathcal{B}$ is then given as follows:

$$\begin{aligned} f(\lambda) = {} & -\left(\frac{\lambda}{a^2} + 1\right)\left(\frac{\lambda}{b^2} + 1\right)\left(\frac{\lambda}{c^2} + 1\right)\left(\lambda + r^2\right) \\ & + \frac{x_c^2}{a^2}\left(\frac{\lambda}{b^2} + 1\right)\left(\frac{\lambda}{c^2} + 1\right)\lambda + \frac{y_c^2}{b^2}\left(\frac{\lambda}{a^2} + 1\right)\left(\frac{\lambda}{c^2} + 1\right)\lambda \\ & + \frac{z_c^2}{c^2}\left(\frac{\lambda}{a^2} + 1\right)\left(\frac{\lambda}{b^2} + 1\right)\lambda. \end{aligned} \tag{3}$$

An inspection of the above expression yields the following lemma.

Lemma 1. *Assuming $0 < a < b < c$, we have*

(1) $f(0) < 0$;
(2) $f(-a^2) < 0$ *if* $x_c \neq 0$, *and* $f(-a^2) = 0$ *if* $x_c = 0$;
(3) $f(-b^2) > 0$ *if* $y_c \neq 0$, *and* $f(-b^2) = 0$ *if* $y_c = 0$;
(4) $f(-c^2) < 0$ *if* $z_c \neq 0$, *and* $f(-c^2) = 0$ *if* $z_c = 0$.

From Lemma 1, and considering the special case of $a = b$ or $b = c$, it is easy to prove:

Theorem 2. *Suppose* $0 < a \leqslant b \leqslant c$. *Then* $f(\lambda) = 0$ *has at least two negative roots in* $[-c^2, -a^2]$, *counting multiplicity. There is a real root in* $[-c^2, -b^2]$ *and there is also a real root in* $[-b^2, -a^2]$.

Lemma 3. *A nonconstant common factor of all the* 3×3 *minors of* $\lambda A + B$ *can only be* $\lambda + a^2$, $\lambda + b^2$ *or* $\lambda + c^2$.

Proof. It suffices to note that the first minor, $\det(\lambda A + B)(1, 2, 3|1, 2, 3)$, is

$$\left(\frac{\lambda}{a^2} + 1\right)\left(\frac{\lambda}{b^2} + 1\right)\left(\frac{\lambda}{c^2} + 1\right). \qquad \square$$

3. Separation of two ellipsoids

We first consider a necessary condition for the separation of two ellipsoids.

Theorem 4. *If* $\mathcal{A}$ *and* $\mathcal{B}$ *are separated, then* $f(\lambda) = 0$ *has two distinct positive roots.*

The next lemma will be used in the proof.

Lemma 5. *If* $f(\lambda) = 0$ *has a positive double root, then* $\mathcal{A}$ *and* $\mathcal{B}$ *have a real touching point.*

Proof. Let $\lambda_0 > 0$ be a positive double root of $f(\lambda) = 0$. By Lemma 3, λ_0 is not a common zero of all the first 3×3 minors of $\lambda A + B$. Hence, the matrix $\lambda_0 A + B$ has rank 3 and its null space, $\mathrm{Ker}[\lambda_0 A + B]$, has dimension 1. Further, $\lambda A + B = A(\lambda I - (-A^{-1}B))$, and thus λ_0 is an eigenvalue of $-A^{-1}B$ with multiplicity 2, and the null space $\mathrm{Ker}[\lambda_0 I + A^{-1}B]$ has dimension 1. By the Jordan canonical form, there are a real eigenvector X_0 and a generalized eigenvector X_1 of $-A^{-1}B$ (see Appendix B of (Strang, 1988)) such that

$$\left(A^{-1}B\right)X_0 = -\lambda_0 X_0 \quad \text{and} \quad \left(A^{-1}B\right)X_1 = -\lambda_0 X_1 + X_0,$$

or, equivalently,

$$\left(\lambda_0 I + A^{-1}B\right)X_0 = 0 \quad \text{and} \quad \left(\lambda_0 I + A^{-1}B\right)X_1 = X_0,$$

which implies that $(\lambda_0 I + A^{-1}B)^2 X_1 = 0$. Since A and B are symmetric,

$$X_0^{\mathrm{T}} A X_0 = X_1^{\mathrm{T}} A\left(\lambda_0 I + A^{-1}B\right)^2 X_1 = 0.$$

Consequently, X_0 is a point on $\mathcal{A}$. Note that X_0 is also a point on $\mathcal{B}$ since $X_0^{\mathrm{T}} B X_0 = X_0^{\mathrm{T}}(\lambda_0 A + B)X_0 = X_0^{\mathrm{T}} A(\lambda_0 I + A^{-1}B)X_0 = 0$.

The tangent planes of $\mathcal{A}$ and $\mathcal{B}$ at X_0 are $X^{\mathrm{T}} A X_0 = 0$ and $X^{\mathrm{T}} B X_0 = 0$, respectively. Since $(\lambda_0 I + A^{-1}B)X_0 = 0$, it follows that $-\lambda_0 A X_0 = B X_0$; in other words, the two tangent planes are identical. Hence $\mathcal{A}$ and $\mathcal{B}$ have a real touching point at X_0. $\square$

Proof of Theorem 4. Consider a sphere $\mathcal{B}_0$ with radius $r > 0$ and center $(a + r + 1, 0, 0)$. Clearly, $\mathcal{A}$ and $\mathcal{B}_0$ are separated. A simple calculation shows that the characteristic equation $f_0(\lambda) = 0$ of $\mathcal{A}$ and $\mathcal{B}_0$ has two negative roots, $-b^2$ and $-c^2$, and two distinct positive roots.

Given a sphere $\mathcal{B}$ of radius r, separated from $\mathcal{A}$, it is possible to move a sphere $\mathcal{B}(t)$, $t \in [0, 1]$, of radius r from $\mathcal{B}_0$ to $\mathcal{B}$, while $\mathcal{B}(t)$ remaining out of contact with $\mathcal{A}$ for any $t \in [0, 1]$: here $\mathcal{B}(0)$ is $\mathcal{B}_0$ and $\mathcal{B}(1)$ is $\mathcal{B}$. Let $f(\lambda; t) = 0$ denote the characteristic equation of the ellipsoid $\mathcal{A}$ and the moving sphere $\mathcal{B}(t)$. We shall show that, for each $t \in [0, 1]$, the characteristic equation $f(\lambda; t) = 0$ has two distinct positive roots. The proof will then follow from the case where $t = 1$.

Recall the following result on the continuity of the roots of a polynomial (see (Bhatia, 1997)): *Let $a_j(t)$, $1 \leqslant j \leqslant n$, be continuous complex-valued functions defined on an interval $\mathcal{I}$. Then there exist continuous complex-valued functions $\alpha_1(t), \ldots, \alpha_n(t)$ which, for each $t \in \mathcal{I}$, constitute the roots of the polynomial equation $\lambda^n - a_1(t)\lambda^{n-1} + \cdots + (-1)^n a_n(t) = 0$.* This result is applicable to our case ($n = 4$) since the leading coefficient of $f(\lambda; t) = 0$, which is $-(abc)^{-2}$, is a nonzero constant for all t. Let $\alpha_i(t)$, where $i = 1, 2, 3, 4$, be continuous functions that constitute the four roots of $f(\lambda; t) = 0$. Since $f(\lambda; 0) = 0$ has two negative roots and two distinct positive roots, the functions $\alpha_i(t)$ can be labeled such that $\alpha_1(0) \leqslant \alpha_2(0) < 0 < \alpha_3(0) < \alpha_4(0)$. Note that, by Theorem 2, $\alpha_1(t)$ and $\alpha_2(t)$ are real-valued and negative for all $t \in [0, 1]$.

Suppose that $f(\lambda; t_0) = 0$ does not have two distinct positive roots for some $t_0 \in (0, 1]$. We recall that $f(\lambda; 0) = 0$ does have two distinct positive roots, and so we must have one of the following two cases:

(i) Either $\alpha_3(t)$ or $\alpha_4(t)$ changes from a positive root into a non-positive real root through 0 or ∞, without ever becoming imaginary first.

(ii) The values $\alpha_3(t_0)$ and $\alpha_4(t_0)$ are a pair of imaginary conjugate roots.

Case (i) is clearly impossible, since the first and last coefficients of $f(\lambda; t)$ are $\det(A) = -(abc)^{-2} \neq 0$ and $\det(B(t)) = -r^2 \neq 0$, respectively.

As regards case (ii), consider the factorization $f(\lambda; t) = a_0(\lambda - \alpha_1(t))(\lambda - \alpha_2(t))g(\lambda; t)$, where $g(\lambda; t) = (\lambda - \alpha_3(t))(\lambda - \alpha_4(t))$. Let $\Delta(t)$ denote the discriminant of $g(\lambda; t)$: $\Delta(t) = (\alpha_3(t) - \alpha_4(t))^2$, which is a real-valued function of t. Clearly, $\Delta(0) > 0$ and $\Delta(t_0) < 0$. By continuity, $\{t \in [0, 1] \mid \Delta(t) = 0\}$ is nonempty and closed; so there exists a least value of t, $t_{\min}$, such that $\Delta(t_{\min}) = 0$. It follows that $g(\lambda; t_{\min})$ has a double real root $\alpha_3(t_{\min}) = \alpha_4(t_{\min})$. Because of the minimality of $t_{\min}$, if $\alpha_3(t_{\min}) = \alpha_4(t_{\min}) \leqslant 0$ were to hold, we would have case (i), which has been shown to be impossible. On the other hand, if $\alpha_3(t_{\min}) = \alpha_4(t_{\min}) > 0$, then, by Lemma 5, the ellipsoid $\mathcal{A}$ and the sphere $\mathcal{B}(t_{\min})$ touch each other, which contradicts the way that $\mathcal{B}(t)$ is constructed. These contradictions imply that no $t_0 \in [0, 1]$ exists such that $\Delta(t_0) < 0$; in other words, case (ii) is also impossible. Hence, $\Delta(t) > 0$ for all $t \in [0, 1]$. Consequently, $\alpha_3(t)$ and $\alpha_4(t)$ are distinct and positive for any $t \in [0, 1]$. □

The next two lemmas will be used later.

Lemma 6. *If $\mathcal{A}$ and $\mathcal{B}$ have a common interior point, then $f(\lambda) = 0$ has no positive root.*

Proof. Let $X_0 = (x_0, y_0, z_0, 1)^{\mathrm{T}}$ denote a point that is contained in the interiors of both $\mathcal{A}$ and $\mathcal{B}$; i.e., $X_0^{\mathrm{T}} A X_0 < 0$ and $X_0^{\mathrm{T}} B X_0 < 0$. Suppose that $f(\lambda_0) = \det(\lambda_0 A + B) = 0$, for some $\lambda_0 > 0$. Then

$$X_0^{\mathrm{T}}(\lambda_0 A + B)X_0 = \lambda_0 X_0^{\mathrm{T}} A X_0 + X_0^{\mathrm{T}} B X_0 < 0.$$

Now, for an arbitrary direction $X_1 = (x_1, y_1, z_1, 0)^{\mathrm{T}}$, let us consider the line

$$X(t) = X_0 + tX_1 = (x_0 + tx_1, y_0 + ty_1, z_0 + tz_1, 1)^{\mathrm{T}}.$$

The value of

$$X(t)^{\mathrm{T}}(\lambda_0 A + B)X(t) = \lambda_0 X(t)^{\mathrm{T}} A X(t) + X(t)^{\mathrm{T}} B X(t)$$

is negative at $t = 0$, and positive for a sufficiently large $|t|$, since in that case $X(t)$ is outside both $\mathcal{A}$ and $\mathcal{B}$. Thus $X(t)^{\mathrm{T}}(\lambda_0 A + B)X(t)$ vanishes at least twice; in other words, the line $X(t)$ intersects the quadric $X^{\mathrm{T}}(\lambda_0 A + B)X = 0$ at two different real points. Since this is the case for an arbitrary direction X_1, the quadric $X^{\mathrm{T}}(\lambda_0 A + B)X = 0$ must be a closed surface in E^3, which must be an ellipsoid. Thus $\det(\lambda_0 A + B) \neq 0$, since the ellipsoid is nondegenerate. But this contradicts the assumption that λ_0 is a root of $f(\lambda_0) = 0$. Hence, all the real roots of $f(\lambda) = 0$ are negative. □

Lemma 7. *If two ellipsoids touch each other externally, then their characteristic equation has a positive double root.*

Proof. We shall first show that the characteristic equation has a positive root, and then show that this positive root is a double root. Let $\mathcal{A}$: $X^{\mathrm{T}} A X = 0$ and $\mathcal{B}$: $X^{\mathrm{T}} B X = 0$ be two externally tangent ellipsoids. Let X_0 be the tangent point of $\mathcal{A}$ and $\mathcal{B}$. Then $BX_0 = -\lambda_0 A X_0$ for some real value $\lambda_0 \neq 0$, since $\mathcal{A}$ and $\mathcal{B}$ share the same tangent plane at X_0. Thus, $(\lambda_0 A + B)X_0 = 0$; that is, λ_0 is a root of $f(\lambda) = 0$, the characteristic equation of $\mathcal{A}$ and $\mathcal{B}$. Let Y_0 be an interior point of $\mathcal{B}$. Then Y_0 must be outside $\mathcal{A}$, since $\mathcal{A}$ and $\mathcal{B}$ are externally tangent. It follows that $Y_0^{\mathrm{T}} B Y_0 < 0$, and $Y_0^{\mathrm{T}} A Y_0 > 0$. On the other hand, the line through X_0 and Y_0 intersects $\mathcal{A}$ at a point U_0 distinct from X_0, and intersects $\mathcal{B}$ at a point V_0 distinct from X_0. Without loss of generality, we may assume that the last components of X_0, Y_0, U_0, and V_0 are all equal to 1. Then we may express $U_0 = (1 - s)X_0 + sY_0$ for some $s < 0$, and $V_0 = (1 - t)X_0 + tY_0$ for some $t > 1$. Since U_0 is on $\mathcal{A}$,

$$\begin{aligned} 0 = U_0^{\mathrm{T}} A U_0 &= (1 - s)^2 X_0^{\mathrm{T}} A X_0 + 2(1 - s)sY_0^{\mathrm{T}} A X_0 + s^2 Y_0^{\mathrm{T}} A Y_0 \\ &= 2(1 - s)sY_0^{\mathrm{T}} A X_0 + s^2 Y_0^{\mathrm{T}} A Y_0. \end{aligned}$$

Hence, $Y_0^{\mathrm{T}} A X_0 = -sY_0^{\mathrm{T}} A Y_0/[2(1 - s)] > 0$; similarly, we can show that $Y_0^{\mathrm{T}} B X_0 = -tY_0^{\mathrm{T}} B Y_0/[2(1 - t)] < 0$. Since $(\lambda_0 A + B)X_0 = 0$, we obtain

$$0 = Y_0^{\mathrm{T}}(\lambda_0 A + B)X_0 = \lambda_0 Y_0^{\mathrm{T}} A X_0 + Y_0^{\mathrm{T}} B X_0.$$

It follows that $\lambda_0 = -Y_0^{\mathrm{T}} B X_0 / Y_0^{\mathrm{T}} A X_0 > 0$.

Recall from Eq. (3) that the leading and last coefficients of $f(\lambda) = 0$ are $-(abc)^{-2}$ and $-r^2$, respectively. Thus the product of all the four roots of $f(\lambda) = 0$ is $(abcr)^2 > 0$. Since there are two negative roots of $f(\lambda) = 0$, and moreover $\lambda_0 > 0$, the fourth root λ_1 must be positive. Now we are going to show by contradiction that $\lambda_1 = \lambda_0$, and consequently that λ_0 is a double root of $f(\lambda) = 0$.

Suppose $\lambda_1 \neq \lambda_0$. Then the real eigenvector X_1 associated with λ_1 is linearly independent from X_0, i.e., X_1 satisfies $(\lambda_1 A + B)X_1 = 0$, and X_1 and X_0 are distinct points. From the equations $(\lambda_0 A + B)X_0 = 0$ and $(\lambda_1 A + B)X_1 = 0$, we get $X_0^{\mathrm{T}}(\lambda_0 A + B)X_1 = 0$ and $X_0^{\mathrm{T}}(\lambda_1 A + B)X_1 = 0$, which implies that $X_0^{\mathrm{T}} A X_1 = X_0^{\mathrm{T}} B X_1 = 0$, since $\lambda_0 \neq \lambda_1$. Thus, X_1 is a point on the common tangent plane of $\mathcal{A}$ and $\mathcal{B}$ at X_0, and X_1 is therefore outside both $\mathcal{A}$ and $\mathcal{B}$: that is, $X_1^{\mathrm{T}} A X_1 > 0$ and $X_1^{\mathrm{T}} B X_1 > 0$. From $(\lambda_1 A + B)X_1 = 0$, we obtain

$$0 = X_1^{\mathrm{T}}(\lambda_1 A + B)X_1 = \lambda_1 X_1^{\mathrm{T}} A X_1 + X_1^{\mathrm{T}} B X_1 > 0.$$

This contradiction implies that the two positive roots λ_0 and λ_1 of $f(\lambda) = 0$ cannot be distinct. Hence, $f(\lambda) = 0$ has a positive double root. □

The next result is the main contribution of this paper.

Theorem 8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two ellipsoids with the characteristic equation $f(\lambda) = 0$.*

Claim (1): *$\mathcal{A}$ and $\mathcal{B}$ are separated if and only if $f(\lambda) = 0$ has two distinct positive roots*;

Claim (2): *$\mathcal{A}$ and $\mathcal{B}$ touch each other externally if and only if $f(\lambda) = 0$ has a positive double root.*

Proof. The 'only if' part of claim (1) is proved by Theorem 4. For the 'if' part, suppose that $f(\lambda) = 0$ has two distinct positive roots. Then, by Lemmas 6 and 7, $\mathcal{A}$ and $\mathcal{B}$ do not have a common interior point and do not touch each other externally. Hence, $\mathcal{A}$ and $\mathcal{B}$ are separated.

The 'only if' part of claim (2) is proved by Lemma 7. For the 'if' part, suppose that $f(\lambda) = 0$ has a positive double root. Now, by Lemma 5, $\mathcal{A}$ and $\mathcal{B}$ touch each other. If $\mathcal{A}$ and $\mathcal{B}$ touch each other internally, then $\mathcal{A}$ and $\mathcal{B}$ have common interior points. Further, by Lemma 6, $f(\lambda) = 0$ has no positive root; this is a contradiction. Hence, $\mathcal{A}$ and $\mathcal{B}$ touch each other externally. □

4. Examples

Three examples of the use of Theorem 8 are presented in this section to illustrate our results.

Example 1. Consider the sphere $\mathcal{A}$: $x^2 + y^2 + z^2 - 25 = 0$ and the ellipsoid $\mathcal{B}$: $(x - 9)^2/9 + y^2/4 + z^2/16 - 1 = 0$. The four roots of the characteristic equation are -6.25, -1.5625, 0.60111, and 4.6211. Since there are two distinct positive roots, $\mathcal{A}$ and $\mathcal{B}$ are separated.

Example 2. Consider the sphere $\mathcal{A}$: $x^2 + y^2 + z^2 - 25 = 0$ and the ellipsoid $\mathcal{B}$: $(x - 6)^2/9 + y^2/4 + z^2/16 - 1 = 0$. The four roots of the characteristic equation are -6.25, -1.5625, $0.1111 \pm 1.663\mathrm{i}$. Since there are no positive roots, $\mathcal{A}$ and $\mathcal{B}$ overlap.

Example 3. Consider the sphere $\mathcal{A}$: $x^2 + y^2 + z^2 - 4 = 0$ and the ellipsoid $\mathcal{B}$: $(x - 8)^2/25 + y^2/4 + z^2/4 - 1 = 0$. The four roots of the characteristic equation are -1.0, -1.0, 0.12554 and 1.2745. Since there are two distinct positive roots, $\mathcal{A}$ and $\mathcal{B}$ are separated. Note that the negative double root -1 indicates that the intersection curve of $\mathcal{A}$ and $\mathcal{B}$ is degenerate in the projective complex space; however, $\mathcal{A}$ and $\mathcal{B}$ do not touch each other.

5. Conclusions

We have presented a necessary and sufficient condition for detecting the intersection of two ellipsoids: their characteristic equation has positive roots if and only if the ellipsoids do not have common interior points. To prove this result for two ellipsoids, an affine transformation is applied to convert one ellipsoid into a canonical form and the other one into a sphere. However, the same condition can be obtained from the characteristic equation of the two original ellipsoids, since affine transformations do not change the roots of the characteristic equation. Hence, an algorithm for testing whether ellipsoids are separated using the above condition does not need to perform an affine transformation.

To apply Theorem 8, the equations $\mathcal{A}$: $X^{\mathrm{T}}AX = 0$ and $\mathcal{B}$: $X^{\mathrm{T}}BX = 0$ must be normalized so that $X_0^{\mathrm{T}}AX_0 < 0$ and $X_0^{\mathrm{T}}AX_0 < 0$, for the interiors of $\mathcal{A}$ and $\mathcal{B}$, respectively. Under this assumption, the two equations can be converted by an affine transformation into the equations (1) and (2), with some positive proportional constants. Since these positive constants do not change the signs of the roots of the characteristic equation, Theorem 8 still holds irrespective of their particular values.

The numerical behavior of an algorithm based on the condition presented here is also of importance; but such a study is beyond the scope of this paper. Furthermore, when two ellipsoids are in motion, and possibly also undergoing some smooth deformation, it is appropriate to consider the zero-set of $f(\lambda; t) = 0$. These issues will be discussed in future work.

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