Quantum Superposition of Communication Lines

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Quantum information has proven to enhance the transmission of information in various senarios. By far, most research studies have focused on the usage of quantum superposition of messages while the superposition of communication channels is rarely explored. This project will investigate the superposition of quantum channels and research on how it could be used to enhance communication. The report will first propose a framework of superposition of quantum channels, and then prove the necessary and sufficient conditions for the validity of the Kraus operators of superposition of channels. An upgrade to the framework will be presented to show that Superposition of quantum channels can be made into a supermap of vacuum extended channels. Further, an example will be given to show superposition of channels can be used to enhance communication capacity under extreme noises. Finally, a comparison between superposition of channels and the case of quantum SWITCH will be made to demonstrate how correlation among channels enhance communication.

CONTENTS

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I. INTRODUCTION

A. Project Overview

Seventy years ago, Shannon established the field of Information Theory in which classical physics is adopted as the framework for communication: classical bits act as the carriers of information and they can be transmitted along well-defined communication line. By using quantum systems as carriers of information, quantum information theory have shown great potentials in enhancing transmission of information including increasing transmission rates[1], enabling unconditional security[2], and introducing new means of information transmission[3][4].

Most existing research has been focusing on the quantum superposition of the messages transmitted. However, the possibility of sending the same message along a superposition of different communication channels is not yet fully explored[5]. This project will explore the notion of superposition of quantum channels, how its framework can be defined and what benifits it can bring in enhancing communication.

B. Motivation

This project is primarily motivated by the power demonstrated by the superposition of **order** of communication channels and the fact that superposition of quantum channels is rarely explored.

The superposition of order will be introduced later in section III.

C. Existing Work

A paper[6] was posted on the arXiv in Oct 2018 (later than the start of this project) on coherent control of quantum channel which explored the superposition of quantum channels. The paper emphasized the classical capacity of superposition of quantum channels (coherent control, as called by its authors). It give the description of a specific implementation and calculate constraints of a mathematical object called the transformation matrix. However, the framework they provide is not general because they cannot realize the superposition of quantum channels as a quantum supermap. Furthermore, their comparison between combined channel and indefinite casual order is not fair.

Earlier this year, a paper [7] was out by Giulio and Hlér. It delivers several results that has been independently developed by this project. Further, it extends the framework of this project and shows it is possible to realize superposition of channels as supermap of vacuum extended channels. The later part of this project is an extension to this paper.

D. Summary of the Report

This report will first give a brief introduction to quantum information for those with little background on this area. We will also introduce quantum SWITCH which is known as the superposition of order. Superposition of quantum channels is physically a superposition of trajectories which will be compared to the superposition of orders (the quantum SWITCH). Both schemes can be understood within the most general framework of superposition of channels.

This project proposes a framework for superposition of quantum channels followed by its mathematical characterization. Then, the general framework for superposition of quantum channels from a recent paper by Giulio and Hlér will be introduced. Based on the upgraded framework, we will present the study of classical capacity of superposition of quantum channels by using the specific example of superposition of depolarising channels. A brief discussion on the effect of correlation among channels on classical capacity will be given. Finally, conclusions will be drawn together with possible next step and future direction.

E. Contribution of this Project

In this project I propose an intuitive framework that describes the superposition of quantum channels. I further prove the necessary and sufficient conditions for the validity of Kraus operators of the superposition of quantum channels. After the the appearance of the paper [7], I collaborate with one of the author Hlér to extend the work. I wrote the code for numerical analysis for a fair comparison of the quantum SWITCH (superposition of orders) with a general superposition of channels. Results shows that the quantum switch (superposition of orders) does have higher capacity compare to superposition of channels. Finally, we briefly study how correlation between quantum channels can enhance communication.

Some of the first part of the results on the general framework are developed simultaneously in the paper [7]. The second part of the results comparing the superposition of channels with the quantum SWITCH will be the topic of an upcoming paper by Hlér, myself and Giulio.

II. A BRIEF INTRODUCTION TO QUANTUM INFORMATION

This section gives a brief introduction of quantum information and computation to those with little or no background to quantum information. **Unless specified otherwise, all contents in this section are from [8] and COMP3316 lecture notes credit to Prof. Chiribella**.

To understand quantum information, there are three major components: quantum states, quantum evolution and quantum measurement.

A. Qubits and Unitary Operators

1. Bits versus Qubits

In classical information theory, the basic unit is bit. In quantum information theory, the bit is replaced by quantum bit or qubit.

A system with two perfectly distinguishable quantum states $|0\rangle, |1\rangle$ can be realized by different systems. For instance, a two dimensional system can be realised by the polarization of a photon, where $|0\rangle$ represents the horizontal polarization and $|1\rangle$ represents the vertical polarization.

This brings us to the most fundamental difference between between bit and qubit: bit can only be in the state of 0 or 1 but a qubit can be in infinitely many forms.

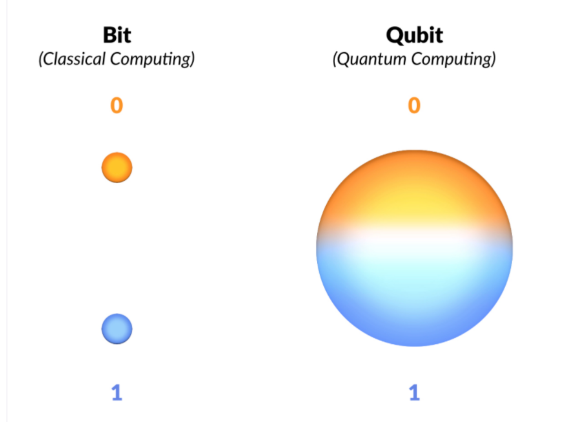

FIG. 1. qubit vs bit[9]

Definition 1 *A qubit can be in any form*

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle \ \alpha, \beta \in C, |\alpha|^2 + |\beta|^2 = 1, |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In fact, the word "superpostion" describes the phenomenon that quantum state can not only be in the perfectly distinguishable state $|0\rangle, |1\rangle$, but also be in the linear combination of these perfectly distinguishable states. As to the previous example, a state $|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ can be thought of as a photon of 45 degrees polarization.

A quantum state that can be represented in above vector form is addressed as **pure state**.

2. Dirac Notation

Dirac notation is a simple way to represent vectors and matrices. In Dirac notation,
$|k\rangle$ is a unit column vector, i.e. $\||k\rangle\| = 1$.
$\langle k|$ is a unit row vector and $\langle k|$ is the complex conjugate transpose of $|k\rangle$ i.e. $\langle k| = |k\rangle^\dagger$

For example, $\langle k|k\rangle$ naturally represents the inner product between the row vector $\langle k|$ and column vector $|k\rangle$. $|k\rangle\langle k|$ is the outer product between a column vector $|k\rangle$ and a row vector $\langle k|$, which is a rank one matrix.

3. Hilbert Space and Orthonormal Basis

A quantum system is associated to a Hilbert space $\mathscr{H}_n = \mathbb{C}^n$.

Definition 2 *A set of vectors $|\phi_n\rangle\}_{n=1}^{d}$ is an orthonormal basis (ONB) for $\mathscr{H} = \mathbb{C}^d$ if and only if*

$$\langle\phi_m|\phi_n\rangle = \delta_{mn}$$

Some commonly used ONBs are computational basis $\{|0\rangle, |1\rangle\}$ and fourier basis $\{|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}, |-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}\}$

4. Unitary Operators

Definition 3 *A linear operator U: $\mathscr{H} \to \mathscr{H}$ is unitary if and only if*

$$U^\dagger U = UU^\dagger = I$$

where I is the identity operator $I|\phi\rangle = |\phi\rangle \forall |\phi\rangle$.

A unitary operator is called a reversible gate because the effect of an unitary can always be undone.

$$U^\dagger(U|\phi\rangle) = U^\dagger U|\phi\rangle = I|\phi\rangle = |\phi\rangle$$

All evolution of pure states can be represented by a unitary operator

$$|\psi\rangle \to U|\psi\rangle$$

5. Basic Measurements

Definition 4 *Basic measurements that can be performed on a quantum sytem are represented by ONBs on the corresponding Hilbert space.*

Each basic measurement has d possible outcome corresponding to the vectors of the ONB.

*If we measure a state $|\phi\rangle$ on the ONB $\{|\phi_n\rangle\}_{n=1}^{d}$, the probability of obtaining outcome n is given by the **Born rule***

$$p(n||\phi\rangle) = |\langle\phi_n|\phi\rangle|^2$$

B. Composite Quantum System

1. Tensor Product Hilbert Space

When Alice has a qubit in Hilbert space $\mathscr{H}_A$, Bob has a qubit in Hilbert space $\mathscr{H}_B$, $\mathscr{H}_A \simeq \mathscr{H}_B \simeq \mathbb{C}^2$.

The tensor product space $\mathscr{H}_A \otimes \mathscr{H}_B = \mathbb{C}^4$ describes the composite system of Alice and Bob. The tensor product associate every column vector $|\alpha\rangle \in \mathscr{H}_A$ and $|\beta\rangle \in \mathscr{H}_B$ to a vector $|\alpha\rangle \otimes |\beta\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$.

$$|a\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, |b\rangle = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}, |a\rangle \otimes |b\rangle = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix}$$

Similarly for matrices, $A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}, B = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}$

$$A \otimes B = \begin{bmatrix} A_{00}B_{00} & A_{00}B_{01} & A_{01}B_{00} & A_{01}B_{01} \\ A_{00}B_{10} & A_{00}B_{11} & A_{01}B_{10} & A_{01}B_{11} \\ A_{10}B_{00} & A_{10}B_{01} & A_{11}B_{00} & A_{11}B_{01} \\ A_{10}B_{10} & A_{10}B_{11} & A_{11}B_{10} & A_{11}B_{11} \end{bmatrix}$$

One can verify some useful formulas for tensor product:

1. $A \otimes kB = kA \otimes B = k(A \otimes B), k \in \mathbb{C}$
2. $(A \otimes B)(C \otimes D) = AC \otimes BD$
3. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$

Definition 5 *Composite System*

Let $\mathscr{H}_A$ and $\mathscr{H}_B$ be the Hilbert space of two quantum system A and B, the Hilbert space of composite system AB is $\mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$.

If A undergoes reversible (unitary) gate U_A and system B undergoes reversible (unitary) gate U_B, then system AB undergoes the reversible gate $U_A \otimes U_B$.

2. Entangled State

Note that not all state in $\mathbb{C}$ can be written into the form of $|a\rangle \otimes |b\rangle$. Consider the state

$$|\Phi^+\rangle = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}$$

Such a state cannot be seen as the product of two state, i.e. it is impossible to find two states $|a\rangle$ and $|b\rangle$ such that $|a\rangle \otimes |b\rangle = |\Phi^+\rangle$. Such a state is called **entangled state**.

C. General Quantum States

Suppose we have a qubit but we do not know its state. We only know it is $|0\rangle$ with probability p and $|+\rangle$ with probability (1-p).

Now we want to measure the state on some new basis $\{|a_0\rangle, |a_1\rangle\}$. Then the probability of getting 0 and 1 respectively are

$$\begin{aligned} p(0) &= p|\langle a_0|0\rangle|^2 + (1-p)|\langle a_0|+\rangle|^2 \\ &= p\langle a_0|0\rangle\langle 0|a_0\rangle + (1-p)\langle a_0|+\rangle\langle +|a_0\rangle \\ &= \langle a_0[p|0\rangle\langle 0| + (1-p)|+\rangle\langle +|]|a_0\rangle \end{aligned}$$

If we define $\rho = p|0\rangle\langle 0| + (1-p)|+\rangle\langle +|$,

$$p(1) = \langle a_1|\rho|a_1\rangle$$

Since $\{|a_0\rangle, |a_1\rangle\}$ are arbitrary ONB, this tells that ρ contains all the information we can ever retrieve from the state. It is an equivalence to the original state.

In fact, it is the general form of quantum state called density matrix.

Definition 6 *Suppose a quantum system of dimension d is in the state $|\phi_0\rangle$ with probability p_0, in the state $|\phi_1\rangle$ with probability p_1, in the state $|\phi_2\rangle$ with probability p_2, and so on. Then, the state of the system can be represented by a density matrix*

$$\rho = \sum_{i=0}^{d-1} p_i |\phi_i\rangle\langle\phi_i|$$

Density matrix has two very important properties:

$$1. \langle\phi|\rho|\phi\rangle \geq 0$$

This can be interepeted as the probability of measurement is always non-negative.

$$2. \operatorname{Tr}[\rho] = 1$$

This can be interpreted as the sum of probability of different outcome when the system is measured in computation basis is 1.

1. Partial Trace

Consider a composite system AB, made of system A and system B. In general, the state of the system can be described by a density matrix ρ_{AB}. A natural question to ask is what is the state is the system A in.

Suppose Alice and Bob prepare to measure the system A and B using ONB $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ seperately. The equivalent measurement on the composite system is $\{|a_i\rangle \otimes |b_i\rangle\}$. One can show that probability of Alice's outcome is independent from that of Bob (which is the famous no-signaling theorem). From this, we can define the state on Alice system alone:

Definition 7 *If the composite system AB is in the state ρ_{AB}, then the state of the system A alone, (**the marginal state**) is*

$$\rho_A = \operatorname{Tr}_B[\rho_{AB}]$$

$$\operatorname{Tr}_B[\rho_{AB}] = \sum_{n=0}^{d_B-1} (I_A \otimes \langle n|)\rho_{AB}(I_A \otimes |n\rangle)$$

D. Purification

Suppose there is the system of A and B are in the pure state

$$|\psi\rangle = \sum_i \sqrt{p_i}|a_i\rangle \otimes |i\rangle$$

One can verify the marginal state of system A is

$$\rho_A = \sum_i p_i |a_i\rangle\langle a_i|$$

From this simple calculation, we can have two interpretations of the density matrix $\sum_i \sqrt{p_i}|a_i\rangle\langle a_i|$.

1. System A is in one of the state $|a_i\rangle$, but we do not know which one. In this situation, the density matrix describes our incomplete knowledge of the sytem.
2. System A belong to the system AB which we have maximal knowledege of: we know AB is in pure state $|\psi\rangle$. Our ignorance of system A comes from the fact that B is discarded. It is like a partion of a puzzle is thrown away and we can only guess what the original picture is.

Definition 8 *For every density matrix ρ of system A, there exists a system B and a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that*

$$\rho = \mathrm{Tr}_A[|\psi\rangle\langle\psi|]$$

The pure state $|\psi\rangle$ is called a purification of ρ and system B is called a purifying system.

E. General Quantum Process

We have introduced unitary matrices for transforming pure states. Similarly, the evolution of a density matrix ρ under the unitary gate U is given by

$$U(\rho) = U\rho U^\dagger$$

According to quantum theory, the basic operations a machine can implement is

1. combining system A with another system B, initialized in some fixed state σ

$$\mathscr{C}_1(\rho) = \rho \otimes \sigma$$

2. letting systems A and B evolve through a unitary gate U_{AB}

$$\mathscr{C}_2(\rho_{AB\to A'B'}) = U_{AB\to A'B'}\rho_{AB\to A'B'}U^\dagger_{AB\to A'B'}$$

3. discarding system B

$$\mathrm{Tr}_{B'}[\rho_{AB}]$$

Combining the three operations together,

Definition 9 *Let $\mathscr{C}$ be a map transforming $d_A \times d_A$ matrices into $d_{A'} \times d_{A'}$. The map $\mathscr{C}$ is called a quantum chanel if and only if it can be decomposed into:*

$$C(\rho) = \mathrm{Tr}_B[U_{AB}(\rho \otimes \sigma)U^\dagger_{AB}]$$

From this definition, it can be further derived that all quantum channels have a simple mathematical representation, the **Kraus Operators**.

Definition 10 *Let $\mathscr{C}$ be a quantum channel transforming $d_A \times d_A$ matrices into $d_{A'} \times d_{A'}$. Then, $\mathscr{C}$ can be written as*

$$\mathscr{C}(\rho) = \sum_{i=0}^{K-1} C_i \rho C_i^\dagger$$

where $\{C_i\}$ is a set of $d_{A'} \times d_A$ matrices satisfying:

$$\sum_{i=0}^{K-1} C_i^\dagger C_i = I_A$$

where $\{C_i\}$ are linear operators called **Kraus operators**.

The Kraus Operators can be extracted from the unitary mentioned in basic step, and vice versa, a set of Kraus Opertors can be used to constuct such an unitary which is called the **Stinespring Dilation** of the channel.

1. *Quantum Supermap*

A quantum channel is sometimes called a map from one Hilbert space to another. A supermap is an operation that takes in multiple maps as inputs and gives a map as output. An anology of supermap will be higher order function.

F. General Quantum Measurement

To obtain information of an unknown quantum state, quantum measurement has to be performed. In previous subsection, we have introduced basic measurements by ONBs. Now we shall introduce general quantum measurement.

A general quantum measurement is characterized by POVM (positive-operator valued measure).

Definition 11 *A set of $d \times d$ matrices is called a POVM is it satisfies the conditions*

1. *Positivity.* $\langle\phi|P_j|\phi\rangle \geq 0, \forall j, \forall|\phi\rangle$, *and*
2. *Normalization.* $\sum_{j=0}^{N-1} P_j = I$

For a measurement carried out with a POVM on state ρ, the possability of each outcome i is

$$p_i = \mathrm{Tr}[P_i\rho]$$

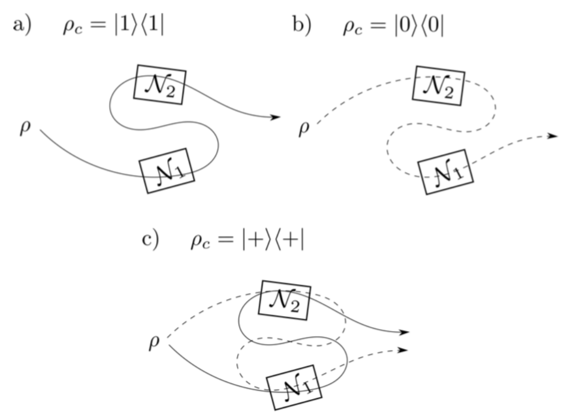

FIG. 2. Quantum Superposition of Order[4]

(a) shows a quantum system pass through N1 and then N2 (b) shows a quantum system pass through N2 and then N1 (c) shows a quantum system pass through N1 and N2 in superposition of order

III. SUPERPOSITION OF ORDER AND QUANTUM SWITCH

Quantum switch, also known as the superposition of order, is the major subject for comparison with superposition of channels in this project. It also serves as a motivation for this project. Therefore, we will briefly introduce what the quantum SWITCH is and what it can do.

1. Superposition of Order and quantum SWITCH

Suppose we have two channels $\mathcal{N}_0$ and $\mathcal{N}_1$. In the classical world, we can have two orders of using the channel sequentially: we can either use $\mathcal{N}_0$ followed by $\mathcal{N}_1$ or $\mathcal{N}_1$ followed by $\mathcal{N}_0$. This classical way of sequential usage is called definite order.

In addition to the definite order, quantum mechanics allows indefinite order. It is possile to use the two quantum channels $\mathcal{N}_0$ and $\mathcal{N}_1$ in the two orders at the same time (shown in Figure 2).

The way to realize this indefinite order is by using a device called quantum SWITCH. The quantum SWITCH is a quantum supermap that takes two quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$ as inputs and creates a new channel that apply the two channels in an order conditioned on an independent control qubit.

Suppose the Kraus operators of $\mathcal{N}_1$ and $\mathcal{N}_2$ are $\{K_i\}$ and $\{K_j\}$ respectively. Then, the overall Kraus operators of the channel obtained from switching $\mathcal{N}_1$ and $\mathcal{N}_2$ is $\{W_{ij}\}$ such that

$$W_{ij} = K_iK_j \otimes |0\rangle\langle 0| + K_jK_i \otimes |1\rangle\langle 1|$$

When acting on a state ρ together with an independent control state ρ_c, the resulting output is

$$S(\mathcal{N}_1, \mathcal{N}_2)(\rho \otimes \rho_c) = \sum_{i,j} W_{ij}(\rho \otimes \rho_c)W_{ij}^\dagger$$

Indeed, when the control state $\rho_c = |0\rangle\langle 0|$, the output state is

$$S(\mathcal{N}_1, \mathcal{N}_2)(\rho\otimes|0\rangle\langle 0|) = \sum_{i,j} K_iK_j(\rho\otimes|0\rangle\langle 0|)K_j^\dagger K_i^\dagger = \mathcal{N}_1\circ\mathcal{N}_2(\rho)\otimes|0\rangle\langle 0|$$

which is equivalent to applying $\mathcal{N}_1$ and $\mathcal{N}_2$ sequentially.
when the control state $\rho_c = |1\rangle\langle 1|$, the output state is

$$S(\mathcal{N}_1, \mathcal{N}_2)(\rho\otimes|1\rangle\langle 1|) = \sum_{i,j} K_jK_i(\rho\otimes|1\rangle\langle 1|)K_i^\dagger K_j^\dagger = \mathcal{N}_2\circ\mathcal{N}_1(\rho)\otimes|1\rangle\langle 1|$$

which is equivalent to applying $\mathcal{N}_2$ and $\mathcal{N}_1$ sequentially.

Therefore, we can see that quantum SWITCH condition the order of two input channels on the control state.

2. Gain in Classical Capacity

A depolarising channel is channel that will fully randomized a quantum state which cannot be used to transmit any classical inforamation.

$$\mathscr{D}(\rho) = \frac{I}{d}$$

Suprisingly, it is shown in [4](Figure2) that when two depolarising channels are used in superposition of orders, they can actually be used to transmit some amount of information.

Such a result encourages one to wonder what kind of power superposition of channels could bring. This project will explore the property of the superposition of quantum channels and how it might be used as a resource for communication.

The superposition of order can be viewed as a superposition of channels in **time**[7]. The superposition of quantum channels that will be explored by this project is in fact a superposition of channels in **space**[7].

Start from next section, we will present the results of this project.

IV. PROPOSED FRAMEWORK FOR SUPERPOSITION OF CHANNELS

This section will propose a general framework describing the superposition of two quantum channels (Figure 3).

A. Defined Behaviour of Superposition of Channels

Suppose we are given two quantum channels $\mathscr{N}_0$ and $\mathscr{N}_1$ which are put on two different paths. Our goal is to control the path the input message take by a control state.

Formally, we denote the Hilbert space of the input message to be $\mathscr{H}_M$ and Hilbert space of a *control state* to be $\mathscr{H}_c$.

Definition 12 *A channel $\mathscr{C} : \mathsf{St}(\mathscr{H}_c \otimes \mathscr{H}_M) \to \mathsf{St}(\mathscr{H}_c \otimes \mathscr{H}_M)$ is a superposition of two channels $\mathscr{N}_0 : \mathsf{St}(\mathscr{H}_M) \to \mathsf{St}(\mathscr{H}_M)$ and $\mathscr{N}_1 : \mathsf{St}(\mathscr{H}_M) \to \mathsf{St}(\mathscr{H}_M)$, if there exist an ONB$\{|0\rangle, |1\rangle\}$ in $\mathscr{H}_c$ such that*

- *When the control state $\rho_c = |0\rangle\langle 0|$*

$$\forall \rho \in \mathscr{H}^M,\ \mathscr{C}(|0\rangle\langle 0| \otimes \rho) = |0\rangle\langle 0| \otimes \mathscr{N}_0(\rho)$$

- *When the control state $\rho_c = |1\rangle\langle 1|$*

$$\forall \rho \in \mathscr{H}^M,\ \mathscr{C}(|1\rangle\langle 1| \otimes \rho) = |1\rangle\langle 1| \otimes \mathscr{N}_1(\rho)$$

B. Notations for Superposition of Channels

The newly constructed channel $\mathscr{C}$ is composed of $\mathscr{N}_0$ *and* $\mathscr{N}_1$ together with a *fixed control state* $\rho_c \in \mathscr{H}_c$ which is **independent** of the message.

For the purpose of this project, we only consider two dimensional control state, which means the state space of the control state $\mathsf{St}(\mathscr{H}_c) = \mathsf{St}(\mathsf{Span}\{|0>, |1\rangle\})$.

When the control state is $|0\rangle\langle 0$ or $|1\rangle\langle 1|$, a path is selected. When the control state $\rho_c = \alpha|0\rangle\langle 0| + \beta|1\rangle\langle 1|, |\alpha|^2 + |\beta|^2 = 1$, the input message takes a linear combination of the two possible paths.

What makes the framework interesting is that the control state can be more than these two pure state or their linear combination. In fact, the control state can be any state in

$$\mathsf{St}(\mathscr{H}_c) = \mathsf{St}(\mathbb{C}^2)$$

which enable the input state to take a superposition of the path.

Throughout this project report, the channel $\mathscr{C}$ that has our above designed property will be addressed as the **combined channel** or **superposition of channels** while its components $\mathscr{N}_0$ *and* $\mathscr{N}_1$ will be called **subchannels**.

The combined channel is a quantum channel and therefore has a Kraus representation, we denote it as:

$$\mathscr{C} : \mathsf{St}(\mathscr{H}^c \otimes \mathscr{H}^M) \to \mathsf{St}(\mathscr{H}^c \otimes \mathscr{H}^M)$$

$$\mathscr{C}(\rho) = \sum_i C_i \rho C_i^\dagger$$

$$\sum_i C_i^\dagger C_i = I$$

where $\{C_i\}$ are the set of Kraus operators of the superposition of channels.

By the above definition, the channel's behaviour is clear when the control state is $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$, but it is not obvious what will be the output state if the control state has off-diagonal element $|0\rangle\langle 1|$ or $|1\rangle\langle 0|$. To thoroughly understand the behaviour of combined channel, we have to develop a rigorous mathematical characterization of the combined channel.

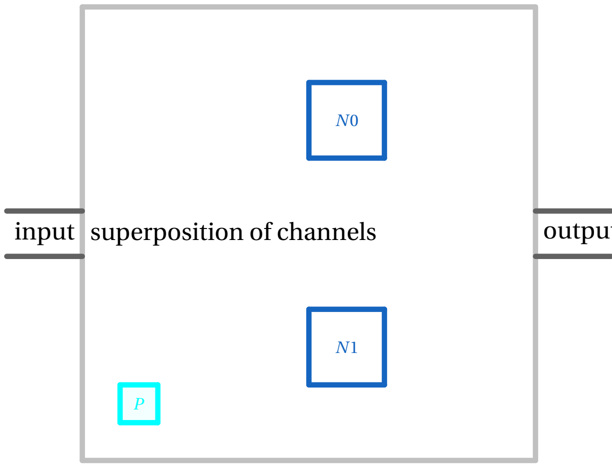

FIG. 3. Composite channel of $\mathscr{N}_0$ *and* $\mathscr{N}_1$

V. MATHEMATICAL CHARACTERIZATION OF SUPERPOSITION OF CHANNELS

To thoroughly understand the property of the superposition of channels, a mathematical characterization needs to be developed. One of the best means is to study the Kraus operators of the superposition of channels which will give us all the information on the channel's behaviour.

In this section, we will prove a necessary and sufficient condition for the validity of a set of Kraus operators of superposition of channels.

To achieve this, we will first define a series of notations and prove some lemmas.

A. Define Hilbert Subspaces

Assume the Hilbert space of the control state to be $\mathscr{H}_c = span\{|0\rangle, |1\rangle\}$, and the Hilbert space of input message to be $\mathscr{H}_M$, define:

$$\mathscr{H}_0 = |0\rangle \otimes \mathscr{H}_M, \mathscr{H}_1 = |1\rangle \otimes \mathscr{H}_M$$

Then, by definition

$$\begin{aligned} \mathscr{H}_c \otimes \mathscr{H}_M = \mathscr{H}_0 \oplus \mathscr{H}_1 \\ \mathscr{H}_0 \perp \mathscr{H}_1 \end{aligned} \tag{1}$$

Lemma 1 *The superposition of channels preserves the input state from* $\mathsf{St}(\mathscr{H}_0)$ *in* $\mathsf{St}(\mathscr{H}_0)$*, and input state from* $\mathsf{St}(\mathscr{H}_1)$ *in* $\mathsf{St}(\mathscr{H}_1)$*.*

$\forall \rho \in \mathsf{St}(\mathscr{H}_0), \mathscr{C}(\rho) \in \mathsf{St}(\mathscr{H}_0)$;
$\forall \rho \in \mathsf{St}(\mathscr{H}_1), \mathscr{C}(\rho) \in \mathsf{St}(\mathscr{H}_1)$

Proof. Without loss of generality, we will prove for $\rho \in \mathsf{St}(\mathscr{H}_0)$.
By definition, $\forall \rho \in \mathsf{St}(\mathscr{H}_0)$, ρ can be written in the form $\rho = |0\rangle\langle 0| \otimes \rho_0$ where $\rho_0 \in \mathsf{St}(\mathscr{H}^M)$.
$\mathscr{C}(\rho) = \mathscr{C}(|0\rangle\langle 0| \otimes \rho_0) = |0\rangle\langle 0| \otimes \mathscr{N}_0(\rho_0) \in \mathsf{St}(\mathscr{H}_0)$

B. Projectors

Let P_0, P_1 be the projectors onto $\mathscr{H}_0, \mathscr{H}_1$.

Lemma 2 *The projectors onto* $\mathscr{H}_0$ *and* $\mathscr{H}_1$ *have following explicit form:*

$$\begin{aligned} P_0 = |0\rangle\langle 0| \otimes I_M \\ P_1 = |1\rangle\langle 1| \otimes I_M \end{aligned} \tag{2}$$

Proof. Without loss of generality, we will prove equation (2) with P_0.
By definition (1), $\forall |\varphi\rangle \in \mathscr{H}^c \otimes \mathscr{H}^M$, $|\varphi\rangle$ can be written as $|\varphi\rangle = |\varphi_0\rangle + |\varphi_1\rangle$ with $|\varphi_0\rangle \in \mathscr{H}_0, |\varphi_1\rangle \in \mathscr{H}_1$

P_0 is the projector onto $\mathscr{H}_0$ if and only if

$P_0(|\varphi_0\rangle + |\varphi_1\rangle) = |\varphi_0\rangle \quad \forall |\varphi_0\rangle \in \mathscr{H}_0, |\varphi_1\rangle \in \mathscr{H}_1$

All $|\varphi_0\rangle \in \mathscr{H}_0$ can be written as $|\varphi_0\rangle = |0\rangle \otimes |\phi_0\rangle$, and similarly $|\varphi_1\rangle = |1\rangle \otimes |\phi_1\rangle$.

Then, we have

$$P_0(|\varphi_0\rangle + |\varphi_1\rangle) = P_0(|0\rangle \otimes |\phi_0\rangle + |1\rangle \otimes |\phi_1\rangle) = |0\rangle \otimes |\phi_0\rangle = |\varphi_0\rangle$$

Which means P_0 is the projector onto $\mathscr{H}_0$.

C. Property of Dagger Channel

Define the dagger channel

$$\mathscr{C}^\dagger(\rho) = \sum_i C_i^\dagger \rho C_i$$

$\mathscr{C}^\dagger(P_0) = \sum_i C_i^\dagger P_0 C_i$ is hermitan. By spectral theorem, it can be decomposed with an orthonormal basis

$$\mathscr{C}^\dagger(P_0) = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|, \ \lambda_i \neq 0, \ |\phi_i\rangle \in \mathscr{H}_0 \oplus \mathscr{H}_1 \tag{3}$$

Lemma 3 *All eigenvectors of $\mathscr{C}^\dagger(P_0)$ are in $\mathscr{H}_0$, all eigenvectors of $\mathscr{C}^\dagger(P_1)$ are in $\mathscr{H}_1$.*

Proof. Without loss of generality, we prove for P_0.

For all $\langle 1| \otimes \langle\psi| \in \mathscr{H}_1$, we have

$$\begin{aligned}\langle 1| \otimes \langle\psi|\mathscr{C}^\dagger(P_0)|1\rangle \otimes |\psi\rangle &= \mathrm{Tr}[\mathscr{C}^\dagger(P_0)|1\rangle\langle 1| \otimes |\psi\rangle\langle\psi|] \\ &= \mathrm{Tr}[P_0\mathscr{C}(|1\rangle\langle 1| \otimes |\psi\rangle\langle\psi|)] \\ &= 0\end{aligned} \tag{4}$$

The third equality holds by lemma 1.

For equation 4 to hold for all vector in $\mathscr{H}_1$, eigenvectors of $\mathscr{C}^\dagger(P_0)$ must lie **entirely** in $\mathscr{H}_0$.

Lemma 3 is proved.

With lemma 3, we are sure $\mathscr{C}^\dagger(P_0)$ can be decomposed with orthonormal eignenvectors in $\mathscr{H}_0$.

$$\mathscr{C}^\dagger(P_0) = \sum_i \lambda_i |0\rangle\langle 0| \otimes |\phi_i\rangle\langle\phi_i| \tag{5}$$

where $\{|0\rangle \otimes |\phi_i\rangle\}$ is an orthonormal basis of $\mathscr{H}_0$

Lemma 4 *(Lindblad [10],rephrased by [11])*
Let $\mathscr{C}^\dagger$ be a identity preserving completely positve map defined as $\mathscr{C}^\dagger(X) = \sum_i C_i^\dagger X C_i$. Then, let $\mathscr{A}$ be an unital algebra. $\mathscr{C}^\dagger$ preserves the elements in $\mathscr{A}$ if and only if all Kraus operators of $\mathscr{C}^\dagger$ are in the commutants of $\mathscr{A}$.

$$C_i \in \mathscr{A}'$$

where

$$\mathscr{A}' = \{B \in \mathscr{L}(\mathscr{H}), [A, B] = 0, \ \forall A \in \mathscr{A}\}$$

Lemma 5

$$\begin{aligned}\mathscr{C}^\dagger(P_0) &= P_0 \\ \mathscr{C}^\dagger(P_1) &= P_1\end{aligned} \tag{6}$$

Proof. Without loss of generality, we will prove equation 6 for P_0.

$\forall$ purestate $|0\rangle\langle 0| \otimes |\varphi\rangle\langle\varphi| \in \mathsf{St}(\mathscr{H}_0)$,

$$\begin{aligned}\mathrm{Tr}[\mathscr{C}^\dagger(P_0)|0\rangle\langle 0| \otimes |\varphi\rangle\langle\varphi|] &= \mathrm{Tr}[P_0\mathscr{C}(|0\rangle\langle 0| \otimes |\varphi\rangle\langle\varphi|)] \\ &= \mathrm{Tr}[\mathscr{C}(|0\rangle\langle 0| \otimes |\varphi\rangle\langle\varphi|)] \\ &= 1\end{aligned} \tag{7}$$

The first equality holds by trace's invariance under cyclic permutation; the second equality holds by lemma 1, and the third equality is true by the trace-preserving property of quantum channels.

Also,

$$\begin{aligned}\mathrm{Tr}[\mathscr{C}^\dagger(P_0)|0\rangle\langle 0| \otimes |\varphi\rangle\langle\varphi|] &= \mathrm{Tr}[\mathscr{C}^\dagger(P_0)(|0\rangle \otimes |\varphi\rangle)(\langle 0| \otimes \langle\varphi|)] \\ &= \langle 0| \otimes \langle\varphi|\mathscr{C}^\dagger(P_0)|0\rangle \otimes |\varphi\rangle\end{aligned} \tag{8}$$

Combine the above two equations,

$$\begin{aligned}\langle 0| \otimes \langle\varphi|\mathscr{C}^\dagger(P_0)|0\rangle \otimes |\varphi\rangle = 1 \\ \forall |0\rangle \otimes |\varphi\rangle \in \mathscr{H}_0 \text{ and } \||0\rangle \otimes |\varphi\rangle\| = 1\end{aligned} \tag{9}$$

By equation (9) and (5), take $\varphi_i = \phi_i$,

$$\begin{aligned}1 &= \langle 0| \otimes \langle\phi_i|\mathscr{C}^\dagger(P_0)|\phi_i\rangle \otimes |0\rangle \\ &= \langle 0| \otimes \langle\phi_i|(\sum_i \lambda_i |0\rangle\langle 0| \otimes |\phi_i\rangle\langle\phi_i|)|\phi_i\rangle \otimes |0\rangle \\ &= \lambda_i\end{aligned} \tag{10}$$

Thus,

$$\mathscr{C}^\dagger(P_0) = \sum_i |0\rangle\langle 0| \otimes |\phi_i\rangle\langle\phi_i| = |0\rangle\langle 0| \otimes I_M = P_0$$

D. Properties of Kraus Operators of Superposition of Channels

Finally, we will prove two very important propositions that will lead us to a necessary and sufficient conditions for the validity of the Kraus operators of superposition of channels.

Proposition 1 *Let $\{C_i\}$ be the set of Kraus operators of the superposition of channels $\mathcal{N}_0, \mathcal{N}_1$; P_0, P_1 are projectors such that $P_0 = |0\rangle\langle 0| \otimes I$, where $|0\rangle\langle 0|$ is the triggering control state for channel $\mathcal{N}_0$, $P_1 = |1\rangle\langle 1| \otimes I$, where $|1\rangle\langle 1|$ is the triggering control state for channel $\mathcal{N}_1$. Then, we have*

$$C_i = P_0 C_i P_0 + P_1 C_i P_1$$

Proof. By proposition 5 and lemma 4

$$\begin{aligned} C_i P_0 = P_0 C_i &\implies P_1 C_i P_0 = 0 \\ C_i P_1 = P_1 C_i &\implies P_0 C_i P_1 = 0 \\ P_0 C_i P_0 + P_1 C_i P_1 &= P_0 C_i + P_1 C_i = C_i \end{aligned} \tag{11}$$

Proposition 2 *Let $\{C_i\}$ be the set of Kraus operators of the superposition of channels $\mathcal{N}_0, \mathcal{N}_1$; P_0, P_1 are projectors such that $P_0 = |0\rangle\langle 0| \otimes I$, where $|0\rangle\langle 0|$ is the triggering control state for channel $\mathcal{N}_0$, $P_1 = |1\rangle\langle 1| \otimes I$, where $|1\rangle\langle 1|$ is the triggering control state for channel $\mathcal{N}_1$. Then, we have*

$$C_i = P_0 C_i P_0 + P_1 C_i P_1$$

$\{\mathrm{Tr}_c[P_0 C_i P_0]\}$ is a valid set of Kraus operators for $\mathcal{N}_0$ and $\{\mathrm{Tr}_c[P_1 C_i P_1]\}$ is a valid set of Kraus operators for $\mathcal{N}_1$.

where $\mathrm{Tr}_c[.]$ is tracing out the control state.

Without loss of generality, we will prove it for $\mathcal{N}_0$. **Proof.** Let $C_i = |0\rangle\langle 0| \otimes K_i + |0\rangle\langle 1| \otimes L_i + |1\rangle\langle 0| \otimes M_i + |1\rangle\langle 1| \otimes N_i$,

$$\begin{aligned} &\sum_i \mathrm{Tr}_c[P_0 C_i P_0] \rho \, \mathrm{Tr}_c[P_0 C_i P_0]^\dagger \\ &= \sum_i K_i \rho K_i^\dagger \\ &= \sum_i \mathrm{Tr}_c[|0\rangle\langle 0| \otimes K_i \rho K_i^\dagger] \\ &= \sum_i \mathrm{Tr}_c[(|0\rangle\langle 0| \otimes K_i)(|0\rangle\langle 0| \otimes \rho)(|0\rangle\langle 0| \otimes K_i^\dagger)] \\ &= \sum_i \mathrm{Tr}_c[P_0 C_i P_0 |0\rangle\langle 0| \otimes \rho P_0 C_i^\dagger P_0] \\ &= \mathrm{Tr}_c[P_0 \sum_i (C_i (|0\rangle\langle 0| \otimes \rho) C_i^\dagger) P_0] \\ &= \mathrm{Tr}_c[P_0 \mathscr{C}(|0\rangle\langle 0| \otimes \rho) P_0] \\ &= \mathrm{Tr}_c[P_0 (|0\rangle\langle 0| \otimes \mathcal{N}_0(\rho)) P_0] \\ &= \mathcal{N}_0(\rho) \end{aligned} \tag{12}$$

E. Necessary and Sufficient Condition for the Validity of Kraus Operators of Superposition of Channels

In fact, we can show that the converse of proposition 1&2 also hold:

Theorem 1 *A channel $\mathscr{C}$ is the combined channel of two subchannels $\mathcal{N}_0, \mathcal{N}_1$ if and only if its Kraus operators satisfy*

1. *$C_i = P_0 C_i P_0 + P_1 C_i P_1$*
2. *$\{\mathrm{Tr}_c[P_0 C_i P_0]\}$ is a valid set of Kraus operators for $\mathcal{N}_0$. $\{\mathrm{Tr}_c[P_1 C_i P_1]\}$ is a valid set of Kraus operators for $\mathcal{N}_1$.*

where $P_0 = |\alpha_0\rangle\langle \alpha_0| \otimes I, P_1 = |\alpha_0\rangle\langle \alpha_1| \otimes I$, $\langle \alpha_0 | \alpha_1 \rangle = 0$, are projectors onto othorgonal Hilbert space $\mathscr{H}_0, \mathscr{H}_1$.

Proof. By proposition 1 and 2, we have proved that the Kraus operators of the superposition of channels have to satisfy the above conditions.

We only need to prove the other direction that when the above two conditions are satisfied for a set $\{C_i\}$, $\{C_i\}$ is a valid set of Kraus operators of superposition of two channels.

By the defined behaviour of superposition of channels 12, we need to prove there is an ONB $|0\rangle, |1\rangle\}$ such that

$$\sum_i C_i (|0\rangle\langle 0| \otimes \rho) C_i^\dagger = |0\rangle\langle 0| \otimes \mathcal{N}_0(\rho)$$

$$\sum_i C_i (|1\rangle\langle 1| \otimes \rho) C_i^\dagger = |1\rangle\langle 1| \otimes \mathcal{N}_1(\rho)$$

Let $|0\rangle\langle 0| = |\alpha_0\rangle\langle \alpha_0|, |1\rangle\langle 1| = |\alpha_1\rangle\langle \alpha_1|$. Without loss of generally, we will prove for the control state being $|0\rangle\langle 0|$. Let the input state be $|0\rangle\langle 0| \otimes \rho$,

By (1), we have

$$\begin{aligned} \mathscr{C}(\rho) &= \sum_i C_i |0\rangle\langle 0| \otimes \rho C_i^\dagger \\ &= \sum_i (P_0 C_i P_0 + P_1 C_i P_1) |0\rangle\langle 0| \otimes \rho (P_0 C_i^\dagger P_0 + P_1 C_i^\dagger P_1) \\ &= \sum_i (P_0 C_i P_0) |0\rangle\langle 0| \otimes \rho (P_0 C_i^\dagger P_0) \\ &= |0\rangle\langle 0| \otimes \mathcal{N}(\rho) \end{aligned}$$

The last equation hold because $\{P_0 C_i P_0\}$ is a set of Kraus operators for $\mathcal{N}_0$ by (2).

Thus, the theorem is proved.

F. Implication of the Theorem

The above theorem gives us *a necessary and sufficient condition* for the validity of the set of Kraus operators for a superposition of channels $\mathscr{C}$ of $\mathscr{N}_0$ and $\mathscr{N}_1$.

By the definition of tensor product, imagine the Kraus operator of C_i are partitioned into four blocks according to the control state.

$$\begin{bmatrix} C_{i00} & C_{i01} \\ C_{i10} & C_{i11} \end{bmatrix}$$

The above theorem states that if $\{C_i\}$ is a set of Kraus operators for the superposition of quantum channels. C_{i01} and C_{i10} must be zero matrices. $\{C_{i00}\}$ is a set of Kraus operators for $\mathscr{N}_0$ and $\{C_{i11}\}$ is a set of Kraus operators for $\mathscr{N}_1$.

The first condition states that only the diagonal block of the combined channel is non-zero. The second one tells that if the two diagonal parts of the Kraus operators of superposition of channels are taken out respectively, they will form complete sets of Kraus operators for the the subchannels.

Given the set of Kraus operators of two channels, the theorem also gives us an recipe on how to create a valid set of Kraus operators for the superposition of two channels.

Given two channels and their corresponding set of Kraus operators are $\mathscr{N}_0, \{K_i\}$ and $\mathscr{N}_1\{L_j\}$ where $\{K_i\}, \{L_j\}$ is allowed to be any valid set of Kraus operators. Then we can

1. Padding arbitrary many zero matrices into $\{K_i\}$ and $\{L_j\}$ as long as the two set are of same size.
2. Create the set of Kraus operators for the superposition of channels by

$$\{C_{ijk} = \sqrt{p(k|i)}e^{i\theta_k}|0\rangle\langle 0| \otimes K_i + \sqrt{q(k|j)}e^{i\gamma_k}|1\rangle\langle 1| \otimes L_j\} \tag{13}$$

where $p(k|i)$ and $q(k|j)$ are conditional probability distributions. The distributions ensure the sum of squares of the coefficients to be 1.

G. Extra Degrees of Freedom

The superposition of the channel is not fully determined by the two set of Kraus operators of the subchannels. There are extra degrees of freedom lie in the implementation which we can show by following example.

Suppose we have a set of Kraus operators in the form of equation (13) (ignore the relative phase for simplicity), and check output involving non-zero off-diagonal control:

$$\begin{aligned}
\mathscr{C}(|0\rangle\langle 1| \otimes \rho) &= \sum_i (\sqrt{p(k|i)}|0\rangle\langle 0| \otimes K_i + \sqrt{q(k|j)}|1\rangle\langle 1| \otimes L_j)(|0\rangle\langle 1| \otimes \rho)(\sqrt{p(k|i)}|0\rangle\langle 0| \otimes K_i + \sqrt{q(k|j)}|1\rangle\langle 1| \otimes L_j) \\
&= \sum_i (\sqrt{p(k|i)}|0\rangle\langle 0| \otimes K_i)(|0\rangle\langle 1| \otimes \rho)(\sqrt{q(k|j)}|1\rangle\langle 1| \otimes L_j) \\
&= \sum_i \sqrt{p(k|i)}\sqrt{q(k|j)}|0\rangle\langle 1| \otimes K_i \rho L_j
\end{aligned}$$

From the output we can observe that when the control state contains off-diagonal element, the output of the channel depend on the implementation, i.e. the specific set of Kraus operators and how are they combined.

There is extra degree of freedom that lie neither in $\mathscr{N}_0$ nor in $\mathscr{N}_1$, but in the implementation detail of the superposition of the channel. This point will be better explained in next section when we investigate the physical implementation of the superposition of channels.

VI. PHYSICAL REALIZATION OF COMBINED CHANNEL

In this section, we will give one way to describe the implementation of the superposition of quantum channels and discuss its properties. The result is from [6]. The goal of this section is to give an intuitive view on possible implementation of superposition of channels whereas a general way of actual implementation will be delivered in next section VII.

All unitary gates (reversible transformation) can be realized physically. Therefore, whenever we want to implement a quantum channel, we can always first construct an isometry of the channel. An isometry can then be further extended to an unitary using standard procedure.

A. Isometry for Combined Channel

Definition 13 *An isometry V of the channel $\mathscr{C}$ is a matrix:*

1. $V^\dagger V = I$
2. $V\rho V^\dagger = \mathscr{C}(\rho)$

According to the result of [6], for a superposition of channels $\mathscr{C}$ with subchannels $\mathscr{N}_0, \mathscr{N}_1$ descirbed by two sets of Kraus operators $\{K_i\}, \{L_i\}$, an isometry of it can be formulated in the following way.

$$V = |0\rangle\langle 0| \otimes \sum_i K_i \otimes |i\rangle \otimes |\epsilon_1\rangle + |1\rangle\langle 1| \otimes \sum_j L_j \otimes |\epsilon_0\rangle \otimes |j\rangle$$

where $\{|i\rangle\}, \{|j\rangle\}$ are ONB corresponding to the subscripts of Kraus operators, $|\epsilon_0\rangle, |\epsilon_1\rangle$ are two initial environmental states.

To prove that it satisfies the first condition:

$$\begin{aligned} V^\dagger V &= |0\rangle\langle 0| \otimes \sum_i K_i^\dagger \sum_j K_j \otimes \langle j|i\rangle + \sum_i L_i^\dagger \sum_j L_j \otimes \otimes \langle i|j\rangle \\ &= |0\rangle\langle 0| \otimes \sum_i K_i^\dagger K_i + |1\rangle\langle 1| \otimes \sum_i L_i^\dagger L_i \\ &= I \end{aligned} \tag{14}$$

To prove that it satisfy the second condition that it act on any input state as the original channel:
Assume $\rho_c = a|0\rangle\langle 0| + b|1\rangle\langle 1| + c|0\rangle\langle 1| + d|1\rangle\langle 0|$, $|\epsilon_0\rangle, |\epsilon_1\rangle$ are the initial states of the environments of $\mathscr{N}_0$ and $\mathscr{N}_1$ respectively.

$$\begin{aligned} &\mathrm{Tr}_{env}[V(\rho_c \otimes \rho)V^\dagger] \\ &= \mathrm{Tr}_{env}[V((a|0\rangle\langle 0| + b|1\rangle\langle 1| + c|0\rangle\langle 1| + d|1\rangle\langle 0|) \otimes \rho)V^\dagger] \\ &= a|0\rangle\langle 0| \otimes \mathscr{N}_0(\rho) + b|1\rangle\langle 1| \otimes \mathscr{N}_1(\rho) \\ &+ c|0\rangle\langle 1| \otimes \sum_i \langle \epsilon_0|i\rangle K_i \rho \sum_j \langle j|\epsilon_1\rangle L_j^\dagger \\ &+ d|1\rangle\langle 0| \otimes \sum_i \langle \epsilon_1|i\rangle L_i \rho \sum_j \langle j|\epsilon_0\rangle K_j^\dagger \end{aligned} \tag{15}$$

Indeed, the V proposed is an isometry of the original channel. Now if we further define two transformation matrix $F_0 = \sum_i \langle \epsilon_0|i\rangle K_i, F_1 = \sum_j \langle \epsilon_1|j\rangle L_j$. We can rewrite the evolution of quantum states into:

$$V(\rho) = a|0\rangle\langle 0| \otimes \mathscr{N}_0(\rho) + b|1\rangle\langle 1| \otimes \mathscr{N}_1(\rho) + c|0\rangle\langle 1| \otimes F_0 \rho F_1^\dagger + d|1\rangle\langle 0| \otimes F_1 \rho F_0^\dagger \tag{16}$$

An isometry can always be extended to an Unitary by standard procedure [8] which then can be realized physically.

B. Limitations of the Scheme

From the above output state, we can observe that the output depend not only on the two subchannels and control state, but on the initial environment state as well.

The extra degree of freedom in the environment makes it impossible to realize the superposition of channel as a quantum supermap under this scheme. However, a recent work [7] shows that it is in fact feasible to realize superposition of quantum channels as a supermap of vacuum extended channels which brings us to the next section.

VII. UPGRADED FRAMEWORK

The above scheme for physical realization is not general as it is not a supermap and it's behaviour depend on the initial environments. An upgraded framework was introduced in an extended work [7] by Giulio and Hlér in which the above problems are mitigated. The upgraded framework will be briefly introduced in this section to (1). complete the full picture of superposition of quantum channels, and (2). facilitate the presentation of latest progress of this project.

The intuitive idea was to pack the extra degree of freedom in environment states in last section into *vacuum extention. All the following contents in this section credited to the aforementioned paper.*

A. Vacuum Extension of a Quantum Channel

When a quantum channel is applied upon message, the message state si passing through the channel. When a quantum channel is not in use, nothing is passing through it.

When nothing is passing through a quantum channel, physically we can model its input as being the *vacuum* state.

For a given channel that acts on a state space A $\mathscr{C}$: $\mathsf{St}(A) \to \mathsf{St}(A\prime)$, its vacuum states lie in a sector that is orthogonal to A.

Formally, let $\mathscr{C} : \mathsf{St}(A) \to \mathsf{St}(A\prime)$, $\mathsf{St}(\text{vacuum}) \perp \mathsf{St}(A)$.

The vacuum extended channel $\tilde{\mathscr{C}}$ of channel $\mathscr{C}$ is

$$\tilde{\mathscr{C}} : \mathsf{St}(A \oplus Vac) \to \mathsf{St}(A\prime \oplus Vac)$$

The vacuum extended channel will act on both the original sector of the input space and additionally act on the vacuum sector. For simplicity, in this project, like in the paper [7], we only consider one dimensional vacuum, which means there a unique state $|vac\rangle$ for vacuum.
Then, for any given channel $\mathscr{C}$ with the set of Kraus operators $\{C_i\}$, the Kraus operators of the vacuum extended channel with one dimentional vacuum space is

$$\tilde{C_i} = C_i \oplus \gamma_i |vac\rangle\langle vac|$$

where $\{\gamma_i\}_{i=1}^{r}$ are complex amplitudes satisfying normalisation condition

$$\sum_i |\gamma_i|^2 = 1$$

The C_i are Kraus operators acting on original sector A and $\gamma_i |vac\rangle\langle vac|$ act on the one dimensional vacuum state $|vac\rangle\langle vac|$ which is orthorgonal to A.

The vacuum space and vacuum amplitudes physically exist and are physical properties of the physical devices that are used to realize the quantum channel. Therefore, we will soon show that they are better and more natural ways to pack the extra degree of freedom.

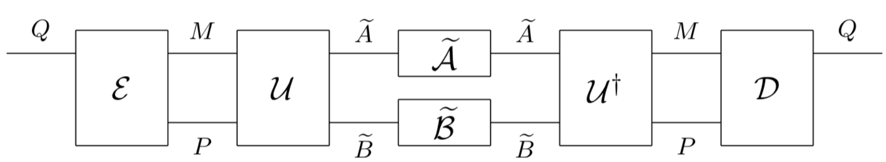

FIG. 4. Quantum Superposition of Independent Channels[7]
Q - the input message state $\in M$
ϵ *- ecoding the input into* $M \otimes P$, *P is the state space of control*
U - the unitary that is isomorphism from $M \otimes P$ *to* $(A \oplus Vac) \otimes (B \oplus Vac)$
D - decoding channel

B. Vacuum Extension of Two Independent Channels

According the results of [7], for any systems A and B, we can construct the vacuum-extended system $\tilde{A} := A \oplus Vac$ and $\tilde{B} := B \oplus Vac$ where Vac is the system of vacuum.

The product system is

$$\tilde{A} \otimes \tilde{B} = (A \otimes B) \oplus (A \otimes Vac) \oplus (Vac \otimes B) \oplus (Vac \otimes Vac)$$

which contains two *one particle sectors* $(A \otimes Vac) \oplus (Vac \otimes B)$.

We can define a superposition of channels A and B by using the one particle part $(A \otimes Vac) \oplus (Vac \otimes B)$. Notice here if we restrict the Vac to be one dimensional, $(A \otimes Vac) \oplus (Vac \otimes B)$ is isomorphic to $M \otimes C$ where M is the system for messages and C is the system for control (the P as denoted in [7]).

The isomorphism guarantees that there exist a unitary U that is an isomorphism from $M \otimes C$ to $(A \otimes Vac) \oplus (Vac \otimes B)$.

$$U(|\alpha\rangle \oplus |\beta\rangle) = (|\alpha\rangle \otimes |vac\rangle) \oplus (|vac\rangle \otimes |\beta\rangle)$$

The Kraus operators for the superposition of channels $S_{\tilde{A},\tilde{B}}$ is:

$$S_{ij} = A_i \beta_j \oplus \alpha_i B_j \tag{17}$$

Another way of writing it is

$$S_{ij} = |0\rangle\langle 0| \otimes A_i \beta_j + |1\rangle\langle 1| \otimes \alpha_i B_j$$

where α_i and β_j are the vacuum amplitudes of the vacuum extended channels of $\mathscr{C}_A$ and $\mathscr{C}_B$.

This expression is essentially the same as the equation (13) derived in this project.

For the superposition of independent channels, the extended framework is operationally equivalent to the framework previously introduced in this project. However, because the degree of freedom from the environment are "packed" into the device (the physical implementation of channel), the superposition is now a *supermap of vacuum extended channels*.

C. Superposition of Copies of Same Channels

By using the expression of Kraus operators in equation (17), it can be derived that the superpostion of two copies of the same channel A, with vacuum extended channels $\tilde{A}$, has following ouput state when the control state is $|+\rangle\langle +|$ or $|-\rangle\langle -|$.

$$\begin{aligned} S_{\tilde{\mathscr{A}},\tilde{\mathscr{A}}}(\rho \otimes |+\rangle\langle +|) &= \sum_{ij} (|0\rangle\langle 0| \otimes A_i \alpha_j + |1\rangle\langle 1| \otimes \alpha_i A_j)(|+\rangle\langle +| \otimes \rho)(|0\rangle\langle 0| \otimes A_i^\dagger \alpha_j + |1\rangle\langle 1| \otimes \alpha_i A_j^\dagger) \\ &= \sum_{ij} \frac{1}{2}\alpha_j \alpha_j^\dagger |0\rangle\langle 0| \otimes A_i \rho A_i^\dagger + \frac{1}{2}\alpha_i \alpha_i^\dagger |1\rangle\langle 1| \otimes A_j \rho A_j^\dagger + \frac{1}{2}\alpha_j \alpha_i^\dagger |0\rangle\langle 1| \otimes A_i \rho A_j^\dagger + \frac{1}{2}\alpha_i \alpha_j^\dagger |1\rangle\langle 0| \otimes A_j \rho A_i^\dagger \\ &= \frac{1}{2}|0\rangle\langle 0| \otimes A(\rho) + \frac{1}{2}|1\rangle\langle 1| \otimes A(\rho) + \sum_{ij} [\frac{1}{2}\alpha_j \alpha_i^\dagger |0\rangle\langle 1| \otimes A_i \rho A_j^\dagger + \frac{1}{2}\alpha_i \alpha_j^\dagger |1\rangle\langle 0| \otimes A_j \rho A_i^\dagger] \end{aligned} \tag{18}$$

which can be rewritten into:

$$\begin{aligned} S_{\tilde{\mathscr{A}},\tilde{\mathscr{A}}}(\rho \otimes |+\rangle\langle +|) = & \frac{A(\rho) + F\rho F^\dagger}{2} \otimes |+\rangle\langle +| \\ & + \frac{A(\rho) - F\rho F^\dagger}{2} \otimes |-\rangle\langle -| \end{aligned} \tag{19}$$

where $F = \bar{\alpha}_i A_i$, $\{A_i\}$ is the set of Kraus operators of A, $\{\alpha_i\}$ is the set of complex vacuum amplitudes.

Similarly we have:

$$S_{\tilde{\mathscr{A}},\tilde{\mathscr{A}}}(\rho \otimes |-\rangle\langle-|) = \frac{A(\rho) - F\rho F^\dagger}{2} \otimes |+\rangle\langle+| + \frac{A(\rho) + F\rho F^\dagger}{2} \otimes |-\rangle\langle-| \tag{20}$$

VIII. CLASSICAL CAPACITY OF SUPERPOSITION OF DEPOLARISING CHANNELS

This section will give an example on how superposition of channels can be used to enhance communication by studying the classical capacity of superposition of depolarising channel.

A. Classical Capacity

Holevo information provides a measure on the amount of classical information that can be transmitted from input party A to output party B through a single use of quantum channel. It is defined as

$$\begin{aligned}\chi(\mathscr{C}) &= max_{\{p_a,\rho_a\}} I(A;B) \\ &= max_{p_j\rho_j}[H(\mathscr{C}(\sum_j p_j\rho_j)) - \sum_j p_j H(\mathscr{C}(\rho_j))]\end{aligned}$$

where $I(A;B)$ is the quantum mutual information, H is the quantum entropy, p_j, ρ_j are the spectral decomposition of the input state ρ such that $\rho = \sum_j p_j\rho_j$, ρ_j are pure states.

B. Superpostion of Completely Eepolarising Channels

A completely depolarizing channel is a quantum channel that will map any input state to maximally mixed state, so it destroys all the information contained in the state.

A depolarising channel of dimension d acting on a input state gives the output:

$$\mathscr{D}(\rho) = \frac{I}{d}$$

It can be represented by uniform randomization over d^2 orthogonal unitary operators.

For this project, we will mainly focus on two dimensional depolarising channel with operators represented as uniform randomization over $\frac{1}{2}\{I, X, Y, Z\}$ where I, X, Y, Z are Pauli matrices defined as $I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, Y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}, Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$.

By equation (24), the superpositon of two independent depolarising channels D has the following output when the control state is $|+\rangle\langle+|$:

$$\begin{aligned}S_{\mathscr{D}_1,\mathscr{D}_2}(\rho \otimes |+\rangle\langle+|) &= \frac{D(\rho) + F\rho F^\dagger}{2} \otimes |+\rangle\langle+| \\ &+ \frac{D(\rho) - F\rho F^\dagger}{2} \otimes |-\rangle\langle-| \\ &= \frac{\frac{I}{2} + F\rho F^\dagger}{2} \otimes |+\rangle\langle+| + \frac{\frac{I}{2} - F\rho F^\dagger}{2} \otimes |-\rangle\langle-|\end{aligned} \tag{21}$$

where $F = \bar{\alpha}_i U_i$ and $\{U_i\}$ is the set of orthogonal unitary operators.

C. Numerical Analysis of Holevo Information of Suprposition of Depolarising Channels

For the simplicity and consistence of the discussion, we will fix the control state to $|+\rangle\langle+|$ in the calculation of classical capacity.

As we have discussed in last section, the output of the superposition of channels depends on the transformation matrix $F = \sum_i \bar{\alpha}_i A_i$ as well, so we need to optimize over all possible F to obtain the Holevo quantity of superposition of channels. Notice that if we choose and fix a specific F, the χ quantity computed will be a lower bound of the Holevo quantity of the channel.

1. Holevo Quantity of $F = \frac{1}{\sqrt{2}}|0\rangle\langle 0|$

In the paper [6], a lower bound of Holevo Information was given by choosing $F = \frac{1}{\sqrt{2}}|0\rangle\langle 0|$ ($F = \frac{1}{\sqrt{2}}\frac{I}{2} + \frac{1}{\sqrt{2}}\frac{Z}{2}$), $p_0 = 0.6, \rho_0 = |0\rangle\langle 0|, p_1 = 0.4, \rho_1 = |1\rangle\langle 1|$, in which case

$$\chi(S_{\mathscr{D}_1,\mathscr{D}_2}(\rho \otimes |+\rangle\langle+|))_{F,rho} \approx 0.16$$

I verified all numerical results in the aforementioned paper numerically and they are correct.

2. Holevo quantity of an F proportional to unitary

Notice that in equation (21), if F is proportional to unitary, we can always apply some unitary to retrieve part of the original state. Suppose $F = \frac{U}{k}$,

$$\begin{aligned} &kU[S_{\mathscr{D}_1,\mathscr{D}_2}(\rho \otimes |+\rangle\langle+|)]kU^\dagger \\ &= \frac{\frac{I}{2} + \frac{\rho}{k^2}}{2} \otimes |+\rangle\langle+| + \frac{\frac{I}{2} - \frac{\rho}{k^2}}{2} \otimes |-\rangle\langle-| \end{aligned}$$

The initial intuition is that when F is proportional to unitary, pre and post processing can help retrieve the original state and thus preserve more classical information.

We can choose $F = \frac{1}{4}(I - iX - iY - iZ)$ such that $2F$ is a unitary. We calculate the holevo quantity for this specific F numerically. However, numerical analysis shows that the Holevo quantity for the above F is only around 0.045 which means it is not the optimal choice of F.

Therefore, the next calculation is an optimisation which includes all possible F.

3. Holevo Quantity Optimised over all F

We have seen two Holevo quantity of superposition of depolarising channels for some specific Fs. Now we would like to optimise Holevo quantity over all possible F and all possible input states.

The chanllenges lie in the fact that there are altogether nine degrees of freedom, three for input states, three for the magnitudes of vacuum amplitudes in constructing F and three for the relative phases of vacuum amplitudes.

Initially the numerical analysis code was written in Python. To cope with the scale of this task, I rewrite in C++ and achieve approximately 10^3 speedup.

The Helevo quantity optimised over all possible F in the form of $F = aI + bX + cY + Z$, and over all input pure states, with precision up to 10% of each variable's range, is around 0.16.

Notice that there are multiple Fs that achieve this bound. We have yet to study the pattern of the F that gives the optimal Holevo quantity by either analyze the F that gives the near optimal numerical results or calculate the analytical form of the Holevo information. This will be done in the forthcoming paper.

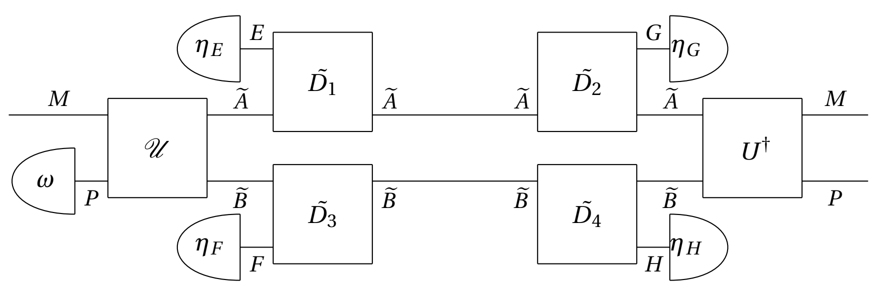

FIG. 5. Superposition of two independent depolarising channels

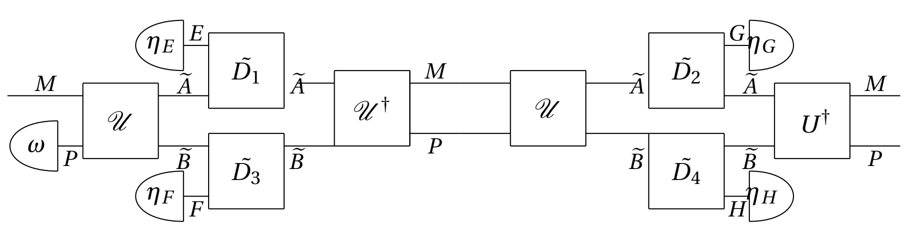

FIG. 6. Sequential Combination of Superpostion of Channels

D. Superposition of Multiple Independent Eepolarising Channels

When we put two independent depolarising channels on each arm of superposition as shown in figure 5, it can be shown that the output of the channel becomes:

$$S_{[\tilde{\mathscr{A}}\tilde{\mathscr{A}}],[\tilde{\mathscr{A}}\tilde{\mathscr{A}}]}(\rho\otimes|+\rangle\langle+|) = \frac{\frac{I}{2}+F^2\rho F^{\dagger 2}}{2}\otimes|+\rangle\langle+| + \frac{\frac{I}{2}-F^2\rho F^{\dagger 2}}{2}\otimes|-\rangle\langle-| \tag{22}$$

$$S_{[\tilde{\mathscr{A}}\tilde{\mathscr{A}}],[\tilde{\mathscr{A}}\tilde{\mathscr{A}}]}(\rho\otimes|-\rangle\langle-|) = \frac{\frac{I}{2}-F^2\rho F^{\dagger 2}}{2}\otimes|+\rangle\langle+| + \frac{\frac{I}{2}+F^2\rho F^{\dagger 2}}{2}\otimes|-\rangle\langle-| \tag{23}$$

There is actually two way of looking at the superposition of two depolarising channels and we verify it they are the same:

1. It can be interpreted as (1). sequentially composite two channels $[\tilde{D_1},\tilde{D_2}]$, $[\tilde{D_3},\tilde{D_4}]$ on each arm respectively. (2). put composited channels into superposition
2. It can also be interpeted as (1). put pairs of channels into superposition $S_{\tilde{D_1},\tilde{D_3}}$, $S_{\tilde{D_2},\tilde{D_3}}$. (2). put $S_{\tilde{D_1},\tilde{D_3}}$, $S_{\tilde{D_2},\tilde{D_3}}$ into sequential composition.

   As shown in figure 6, one can imagine there is a $\mathscr{U}^\dagger$ followed by $\mathscr{U}$ which seperates the superposition of two pairs of channels into sequential combination of superpositions $S_{[\tilde{D_1},\tilde{D_3}]}$ and $S_{[\tilde{D_2},\tilde{D_4}]}$.

The Holevo quantity for the superposition of two pairs of depolarising channels for some specific F are shown in table I. It is apparent that the capacity we get from superposition of two depolarising channels is significantly less than that of one.

F	HQ-one	HQ-two
1/4(I-i(X+Y+Z))	0.049	0.003
1/4(I+X+Y+Z)	0.117	0.018
$1/2\sqrt{2}(I+Z)$	0.161	0.024

TABLE I. Holevo Quantity of superpostion of one of depolarising channel and two pairs of independent depolarising channels

Further, we verify this numerically over all possible F and it can be shown that superposition of two depolarising channels always give less capacity than the case in which only one depolarising channel is put into superposition which lead to the observation in following parts.

E. Superposition of Many Independent Depolarising Channels

If there are N depolarising channels on each arm of superposition, by induction, one can show that the output of superposition of N depolarising channels is

$$S_{[\tilde{\mathscr{A}}\cdots\tilde{\mathscr{A}}],[\tilde{\mathscr{A}}\cdots\tilde{\mathscr{A}}]}(\rho\otimes|+\rangle\langle+|) = \frac{\frac{I}{2}+F^N\rho F^{\dagger N}}{2}\otimes|+\rangle\langle+| + \frac{\frac{I}{2}-F^N\rho F^{\dagger N}}{2}\otimes|-\rangle\langle-| \tag{24}$$

With a help of a lemma that we will prove shortly, we can show

Theorem 2

$$\lim_{N\to\infty} S_{[\tilde{\mathscr{A}}\cdots\tilde{\mathscr{A}}],[\tilde{\mathscr{A}}\cdots\tilde{\mathscr{A}}]}(\rho\otimes|+\rangle\langle+|) = \frac{I}{2}$$

which says the output state will tend to maximally mixed state if there are many depolarising channels, i.e. no information can be transmitted.

The lemma that help to prove to is:

Lemma 6 *For $F = aI + bX + cY + dZ$, $|a|^2+|b|^2+|c|^2+|d|^2 = 1$,*

$$\lim_{N\to\infty} F^N = 0$$

Proof. For $F = \frac{1}{2}(aI + bX + cY + dZ)$, $|a|^2+|b|^2+|c|^2+|d|^2 = 1$,

$$||F||_1 \leq \frac{1}{2}(|a|+|b|+|c|+|d|)$$

where $||.||_1$ is the first norm of matrix which is equivalent to the maximum column sum of absolute values of entries.

Because $|a|^2 + \frac{1}{4} \geq |a|$, and similarly for b, c and d.

$$||F||_1 \leq \frac{1}{2}(|a|+|b|+|c|+|d|) \leq \frac{1}{2}(|a|^2+|b|^2+|c|^2+|d|^2+1) = 1$$

The inequality only hold when $|a| = |b| = |c| = |d| = \frac{1}{2}$.

When $||F||_1 < 1$,

$$\lim_{N\to\infty} F^N = 0$$

.

The only possible form of F that has $||F||_1 = 1$ is $\begin{bmatrix} \frac{1}{2}e^{in} & 0 \\ \frac{1}{2}e^{im} & 0 \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{2}e^{in} & \frac{1}{2}e^{im} \\ 0 & 0 \end{bmatrix}$, in which cases we can verify that $\lim_{N\to\infty} F^N = 0$ as well.

Once we know $\lim_{N\to\infty} F^N = 0$, it is apparent that the theorem hold.

IX. COMPARISON WITH INDEFINITE CASUAL ORDER

A. Comparison of Classical Capacity

In this section, we will revisit a comparison of superposition of channel and quantum SWITCH which first appeared in [6] and show that it is actually not a fair comparison. The paper [6], compares the classical capacity of superposition of one depolarising channel with the case of quantum SWITCH of two depolarising channels and claim that superposition gives a higher classical capacity.

However, as admitted by the author of [6] in his paper, the comparison between coherent control of two depolarizing channel and the indefinite casual order (quantum SWITCH) is not entirely fair. Unlike the case of indefinite casual order, in which the input quantum state pass through depolarizing channel two times, the quantum state only needs to pass through depolarizing channel for one time.

A fair comparison would be between quantum SWITCH of two depolarising channels and superposition of two depolarising channels.

Numerical analysis, optimizing over all F and all input states with percision 10% of each variable's range, shows that the best Holevo quantity we can obtain for superposition of two independent depolarising channels is around 0.024 which is less than the capacity of quantum SWITCH 0.049.
This result agrees with the intuition that the more correlation enabled by quantum SWITCH helps to preserve more information. The study of correlation will be presented in next part of the report.

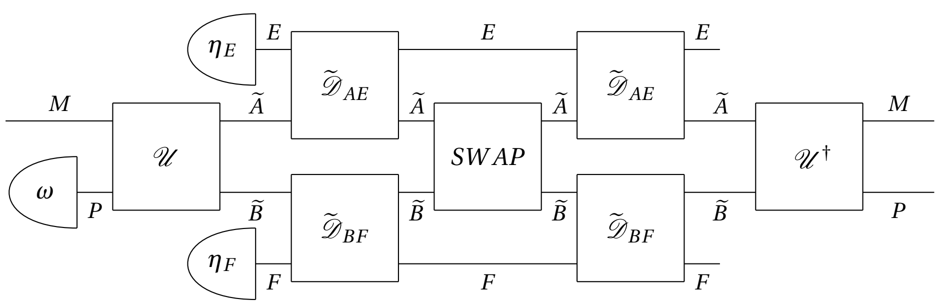

FIG. 7. Superposition of two correlated depolarising channels [7]

B. Correlation of Channels

In this part, we will show that by creating additional correlation among channels, we are able to make the superposition of channels behave like quantum SWITCH.

Quantum SWITCH of two depolarising channels has the set of Kraus operators [4]

$$W_{ij} = U_i U_j \otimes |0\rangle\langle 0| + U_j U_i \otimes |1\rangle\langle 1|$$

where $\{U_i\}$ is a set of orthogonal unitary operators chosen for the specific implementation.

Suppose now we have the superposition of two independent depolarising channels and their Kraus operators are $\{U_i\}, \{U_j\}, \{U_k\}, \{U_l\}$.

The Kraus operators of the superpositon of independent channels are

$$S_{ijkl} = U_i U_j \bar{\alpha}_k \bar{\alpha}_l \otimes |0\rangle\langle 0| + U_k U_l \bar{\alpha}_i \bar{\alpha}_j \otimes |1\rangle\langle 1|$$

where $\{\bar{\alpha}\}$ are the vacuum amplitudes.

If create perfect correlation between i and l, and perfect correlation between j and k, by

1. multiplying S_{ijkl} with $4\delta_{il}\delta_{jk}$, and
2. choose all the vacuum amplitudes to be equal to $\frac{1}{2}$.

$$\begin{aligned} S_{ijkl} 4\delta_{il}\delta_{jk} &= 4(U_i U_j \bar{\alpha}_i \bar{\alpha}_j \otimes |0\rangle\langle 0| + U_j U_i \bar{\alpha}_i \bar{\alpha}_j \otimes |1\rangle\langle 1|) \\ &= U_i U_j \otimes |0\rangle\langle 0| + U_j U_i \otimes |1\rangle\langle 1| \\ &= W_{ij} \end{aligned} \tag{25}$$

Indeed, we can modify the circuit in figure 5 to obtain the circuit in figure 7. Instead of having an independent environment for each channel, environments are shared by pairs of channels. The SWAP in the middle is to swap the messages on the lower arm to the upper arm and vice versa.

A good intuition here is that the consistent environment is an agent for correlated Kraus operators. We can imagine that the Kraus operators are all extracted from the Stinespring dilation U of the channel.

$$V_i = \langle i|\tilde{V_{AE}}|\eta\rangle$$

Therefore consistent environment gives way to correlation.

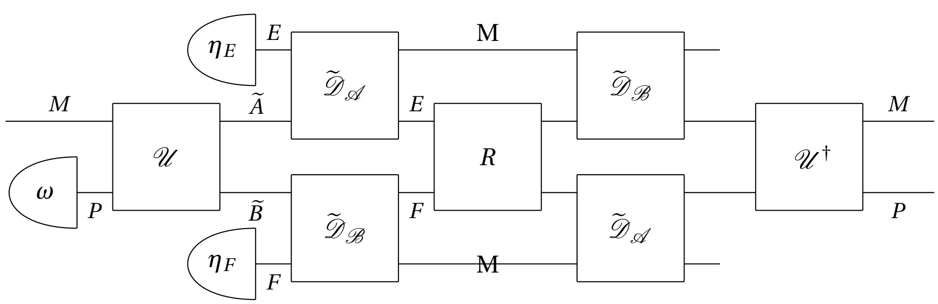

FIG. 8. Convex combination of superposition of channels and quantum switch
Notice here the topology is different from above.
The environmental states $|\eta_E\rangle, |\eta_F\rangle$ flow into the central channel instead of the messages.

C. Convex Combination of Superposition and Quantum SWITCH

Finally, we will give a discussion how correlation among subchannels affect the classical capacity of the overall channel.

1. Create Convex Channel

We have shown that the Holevo information of the quantum SWITCH of two depolarising channels is greater than that of the superposition of two depolarising channels. In last previous part, we have showed that by having perfect correlation between two pairs of channel, we are able to make the superposition of channels behave like the SWITCH which has higher classical capacity.

We raised the question what would be the optimal level of correlation for communicating classical information in this senario. This question can be investigated by putting superposition of channels and the (the equivalence of) quantum SWITCH into convex combination.

We want to construct a channel that will give us an output equivalent to

$$\mathscr{C}_c = \lambda S_{D_1,D2} + (1-\lambda) S_{[D_1 D_2],[D_3 D_4]}$$

where $S_{D_1,D2}$ is the quantum switch (superposition of order) of two depolarising channels, and $S_{[D_1 D_2],[D_3 D_4]}$ is the superposition of two sets of two independent depolarising channels.

This convex channel can indeed be constructed by modifying the topology of the circuit in figure 7 to generate the new channel in figure 8.

In figure 8, the environment E and F together flow into a channel R, where we define R as:

$$R(|\eta_E\rangle \otimes |\eta_F\rangle) \begin{cases} |\eta_E\rangle \otimes |\eta_F\rangle, & \text{with probability} = \lambda \\ |\phi_E\rangle \otimes |\phi_F\rangle, & \text{otherwise} \end{cases} \tag{26}$$

R preserves the environment with probability λ and gives an independent new set of environments $|\phi_E\rangle \otimes |\phi_F\rangle$ otherwise.

Finally, we can verify that with probability λ, it behaves as quantum SWITCH and with probability $(1-\lambda)$ it behaves like superposition of independent channels.

2. Optimal lambda for Holevo information of Convex Channel with Chosen F

We numerically optimize the Holevo quantity of the convex channel with respect to lambda for some chosen F. The results are plotted in figure 9,10,11.

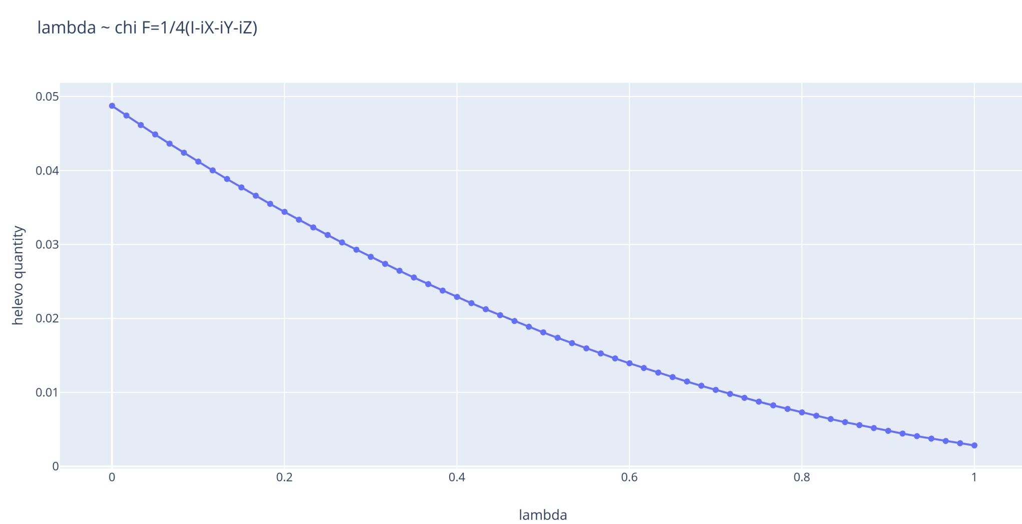

FIG. 9. Holevo quantity ~ lambda for F =1/4(I-iX-iY-iZ)

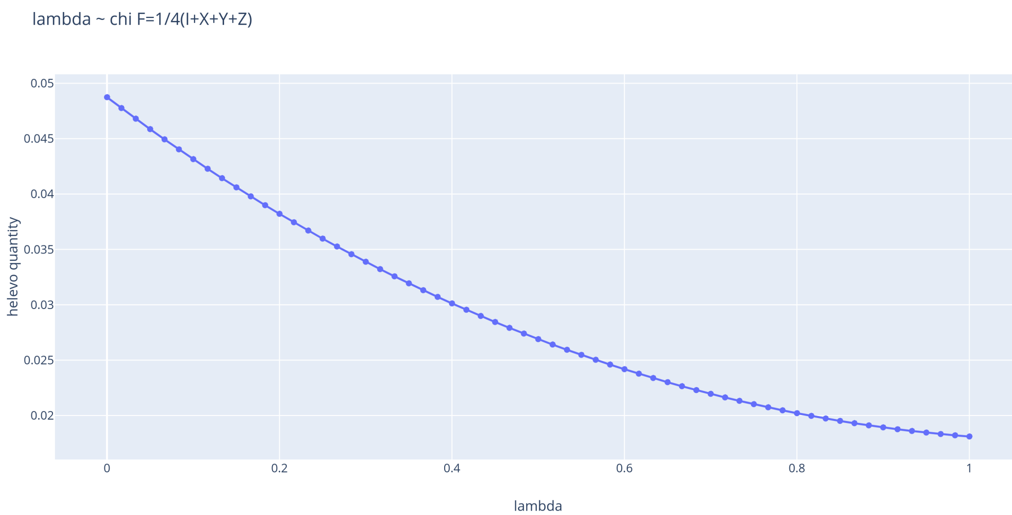

FIG. 10. Holevo quantity ~ lambda for F =1/4(I+X+Y+Z)

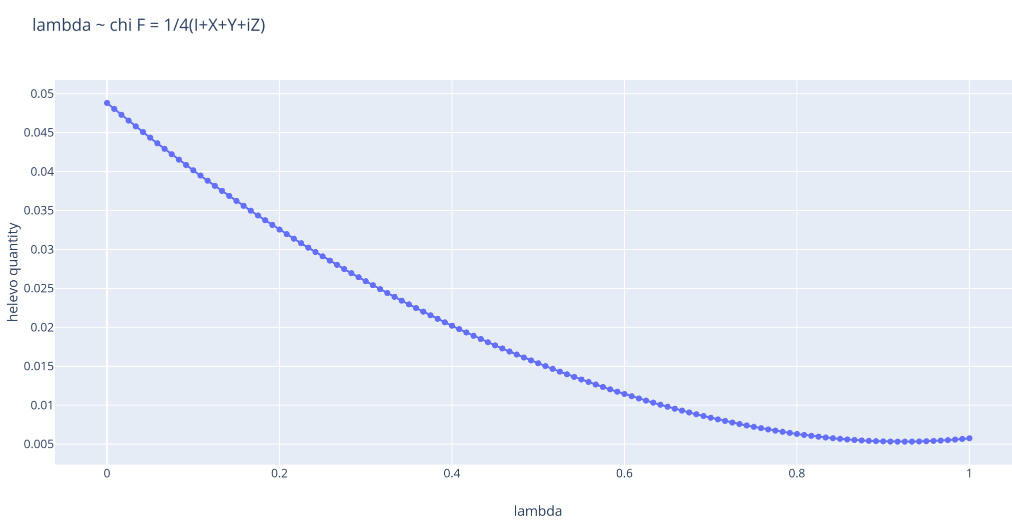

FIG. 11. Holevo quantity ~ lambda for F =1/4(I+X+Y+iZ)

From all three chosen F, the Holevo quantity drops with the increase of λ, i.e. The less proportion of the quantum SWITCH is used, the less correlation we have among channels, and the less the Holevo information.

The relation is non-linear due to the non-linearity of the entropy. Notice here that two of the plots show in figures 9,10 have monotonically decreasing Holevo information with the increase of λ.

However, when $F = \frac{1}{4}(I + X + Y + iZ)$, (shown in figure 11) the convex channel has lowest Holevo quantity at $\lambda \approx 0.93$. This is highly unlikely a result of numerical error (the difference at 4th decimal place but 15 digits of precision is preserved throughout calculation) but rather something interesting for future exploration.

X. CONCLUSION AND FUTURE PLAN

A. Framework of Superposition of Quantum Channels

So far we have proposed a general framework of the superposition of two quantum channels. We have also proved the important theorem that a channel $\mathscr{C}$ is a superposition of two quantum channels $\mathscr{N}_0$ and $\mathscr{N}_1$ if and only if its Kraus operators satisfy:

- $C_i = P_0 C_i P_0 + P_1 C_i P_1$
- $\{\mathrm{Tr}_c[P_0 C_i P_0]\}$ is a set of Kraus operators for $\mathscr{N}_0$ $\{\mathrm{Tr}_c[P_1 C_i P_1]\}$ is a set of Kraus operators for $\mathscr{N}_1$

It is shown that the behaviour of the superposition of channels depend not only on the Kraus operators of the sub-channels but also on the exact implementation (or on the devices use to implement the individual channel).

Different interpretation of the extra degrees of freedom gives rise to different scheme of physical realization. If the extra degrees of freedom is to be packed into the environment, the resulting implementation is environment dependent and is not a supermap of the subchannels. If the extra degrees of freedom is to be packed into vacuum extention of the channel, the superposition is a supermap of the vacuum extended subchannels.

B. Classical Capacity of Superposition of Depolarising Channels

This project has performed numerical analysis of the classical capacity of superposition of depolarising channels. The Holevo information of superposition of one channel is around 0.16 and the Holevo informatino of superposition of two channels is approximately 0.024. It is worth noticing that if a fair comparison between quantum SWITCH and superposition of two channels is carried out, quantum SWITCH has significantly better Holevo information 0.049 compare to 0.024 of superposition of channels.

Another interesting point is that if we put N depolarising channels on each arm of superposition, as N tends to infinity, the classical capacity will eventually drop to zero.

C. Future Direction

In the case of superposition of one depolarising channel, it is yet to explore what kind of F gives rise to the optimal Holevo quantity. Analytical calculation on Holevo information could be carried out or the pattern on the numerical results could be studied.

General correlation among channels and how it can affect channels' capacity could be investigated. This is the topic of a follow-up project and may explain the local minima in the plot of Holevo information vs. lambda in figure 11.

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