Final Report for Final Year Project

A Game-theoretic and Algorithmic Study of the Toll Rates of Hong Kong Road Tunnels

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Abstract

Hong Kong is suffering from great traffic jams, the three crossing-harbor tunnels included. The good news is that the traffic does not evenly distribute among the three tunnels. The western Harbor Tunnel is not fully used up and the Eastern Harbor Tunnel is facing less transportation than the Crossing Harbor Tunnel. By re-allocating traffic properly among the three tunnels, the congestion could be hopefully alleviated or even solved. This report explains the possibility and feasibility of re-allocating the transportation by adjusting the toll rates of the tunnels. By construction of models with or without toll rates, the project proves that toll rate can affect the flow allocation of a road system to a large extent. This project also provides how to find a fixed set of toll rates which ensures a both stable and optimal situation of the road system. Besides that, the project investigates how to dynamically adjust toll rates with the changing of flow of the system and find that equilibrium can be achieved with dynamic pricing strategy. This project offers a relatively thorough theoretical analysis of Hong Kong tunnel system and can hopefully offer a direction for solving the problem.

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I would like to thank Dr. Huang Zhiyi who supervises my final year project and gives me constant help and encouragement.

Then I would like to thank those who have assisted and inspired me during the implementation of the project.

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Abbreviations

Term	Meaning
POA	Price of Anarchy
CHT	Cross-Harbor Tunnel
EHC	Eastern Harbor Crossing
WHC	Western Harbor Crossing

1. Introduction

This section provides an overview of the whole project.

1.1. Background information

Hong Kong has been troubled from traffic jams, especially during peak hours. The cross-harbor tunnels system connecting Kowloon and Hong Kong Island is an epitome of the congested transportation system of Hong Kong.

The cross-harbor tunnels system consists of three tunnels. The Cross-Harbor Tunnel (CHT) carries the largest portion of traffic due to its convenient location and the lowest toll rate. The Eastern Harbor Crossing (EHC) charges the second lowest toll rate and attracts the second largest portion of traffic. Remaining traffic is carried by the Western Harbor Crossing (WHC).

Recently, both the CHT and the EHC are suffering from severe traffic congestion problem. According to the Legislative Council of Hong Kong, the transportation demand of the CHT and the EHC during the peak hours has exceeded their design capacity by 77% and 38% respectively [1]. Meanwhile, the WHC only faces a demand of 90% of its design capacity, indicating that WHC can hold more traffic [1]. Therefore, the congestion of the CHT and the EHC could be hopefully alleviated by part of their traffic being redirected to the WHC.

To achieve this goal, different suggestions have been proposed. Among these proposals, changing toll rates of the three tunnels is considered as an both economic and efficient one.

1.2. Previous Works

Some progress has been achieved in the field of pricing strategy in a congested road network. Back to the 1920s, marginal cost pricing came out, suggesting that each road user should pay for the delay she or he caused for all the users using the same road [2]. Future research shows that the inefficiency caused by the selfish drivers can be contracted by efficient pricing strategy [2].

Apart from the pricing strategy, research on how people will act in a congested road system has also been underway. Wardrop proposed the theory called user equilibrium, suggesting that if every road user knows the overall situation of the road system and chooses the shortest path, the whole system will reach an equilibrium where no one could benefit by only changing their own choice [3]. Later to the 2001, Tim Roughgarden investigated the actions of selfish routing, where every driver acts according to her or his own benefit and calculated how these selfish actions will influence the performance of the whole road system [4].

For the specific congestion problem of the cross-harbor tunnels in Hong Kong, HAI YAN proposed a price strategy of the tunnels by modeling this problem as a two-layer optimization problem [5].

1.3. Scope of the Project

This project tries to model the traffic congestion problem as a congestion game in the field of the Algorithmic Game Theory to lighten the traffic burden of the CHT and the EHC and optimize the performance of the whole cross-harbor tunnels system. In this model, the starting point and the ending point of each tunnel along with possible ways between the pair of points constitute a network. Each game player, known as the network user or driver, seeks to maximize their own profits and cares nothing about the performance of the whole network [4]. How to assign the flow to each edge to minimize the sum of the travel time of each user (referred as total latency in the following part of the paper) will be investigated. This kind of assignment in this project is mainly controlled by the toll rates charged by each tunnel. This project aims at finding a proper pricing strategy to minimize the congestion of the tunnel system.

The significance of the project is two-folded. On the one hand, it provides a detailed and up-to-date model close to the fact which can help understand the problem better. Hong Kong develops at a fast pace, so the models coming up twenty or ten years ago by Yan may not be suitable for the situation now. On the other hand, it could hopefully alleviate the congestion problem at a relatively lower cost. In fact, Hong Kong government has been seeking for different methods of alleviating the congestion problem of the three tunnels. Different suggestions are proposed. Some people suggest that electronic toll booths should take place of manual toll booths so that the waiting time for payment is reduced and the total time needed for passing the tunnel is less [1]. Other people suggest that the road systems on the two endpoints of WHC need further improvement [1]. Then more drivers would be willing to choose the convenient WHC. Both suggestions require huge money investment. Comparatively speaking, changing the pricing system is an economic method.

1.4. Outline

In the following context, the report first introduces some technical concepts related to the project. Then the report elaborates the main methodology adopted by the project, including how to model the problem and the algorithm of calculating pricing strategy. After that, the results of the algorithm will be evaluated. Next, the challenges and possible future work is discussed. Finally, a brief conclusion is offered.

2. Related Concepts and Theorems

In this section, some related concepts and theorems are introduced for better understanding.

2.1. Basic Concepts

2.1.1. Non-cooperative Game

Non-cooperative game is a competition between individual players caring about only their own benefits [6]. This kind of competition usually takes place when the external authority is absent. The model of the cross-harbor tunnel is considered as a non-cooperative game because the Hong Kong government only offers suggestions to drivers about choosing the road.

2.1.2. Potential Game

Potential game describes the game where all players has the same strategy function. The strategy function is called as potential function.

2.1.3. Congestion Game

Congestion game is a game where the payoff of each player is decided by the resource chosen by the player and how many other players choose the same resource. Congestion game is a special kind of potential game. In this paper, the player is the road user and the resource are the tunnels.

2.1.4. Nonatomic Game

For a nonatomic game, the players in the game are endowed in a nonatomic way. In a nonatomic game, a single player has negligible influence over the whole system while the aggregative behavior of a large number of players can change the payoff system.

2.1.5. Nash Equilibrium

Nash equilibrium is a situation where no game player could benefit from changing only his or her own strategy when everything else stays the same in a no-cooperative game [7].

2.1.6. Wardrop Equilibria

Wardrop equilibria were proposed by Wardrop in 1952, which is commonly used for predicting the flow situation in a congested network. The equilibria consist of user equilibrium and system optimal [3].

User equilibrium means that in a congested network, if all users have an overall idea of the whole system, they will choose the shortest road and the system will reach an equilibrium [3]. In this equilibrium, all road with flow has the same passing time. The unused road's passing time is either equal to or larger than the passing time of the used ones [3].

System optimal means that when the road system reaches the user equilibrium, the average travel time is at a minimum [3].

The above two concepts draw a picture about actions of the users and the performance of the whole system when travel time is the only determining factor. The concepts are used as a starting point of the project.

2.1.7. Selfish Routing

Selfish routing in this report indicates the fact that all the drivers choose the best route for themselves and cares noting about the performance of the whole system.

2.1.8. Price of Anarchy (POA)

Price of anarchy refers to the ratio between the performance of the system when all game players choose their own most profitable strategy and the best possible performance of the system. In this paper, the POA refers the ratio between the performance of the system at Nash equilibrium and the system's best possible performance. This ratio is adopted to analyze the results of the project in a quantitative way [4].

2.1.9. Pigou-like Network

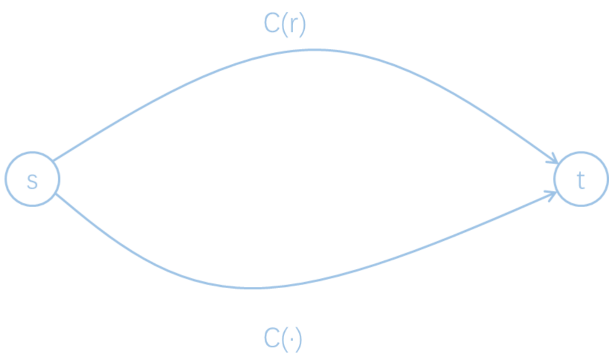

Figure 1: Pigou-like network

As figure 1 shows, a Pigou-like network has:

- ✓ Two vertices, s and t.
- ✓ Two edges from s to t.
- ✓ A traffic rate $r > 0$.
- ✓ A cost function $C(\cdot)$ on the upper edge
- ✓ A cost function everywhere equal to $C(r)$ on the second edge, which is always larger than or equal to $C(\cdot)$. When all the traffic choose the second edge, $C(\cdot)$ equals to $C(r)$

2.2. Important Theories

2.2.1. Existence of Equilibrium

For games with non-cooperative players and a set of payoff functions, at least one equilibrium exists [4].

2.2.2. Tight Bound of POA in Selfish Routing

Among all networks with cost functions, the largest POA exists in a Pigou-like network.

3. Methodology

In this section, how to model the congestion problem will first be discussed. Then the procedures of constructing a scenario for applying the model will be introduced. Afterwards, how to predict and analyze the system's performance in the constructed scenario will be investigated.

3.1. Model of the Problem

The model of the problem is divided into three parts, that is modeling the tunnel system, modeling the user behavior and constructing an application scenario . This project uses the model put forward by Richard Cole in 2003 for reference.

3.1.1. Model of Tunnel System

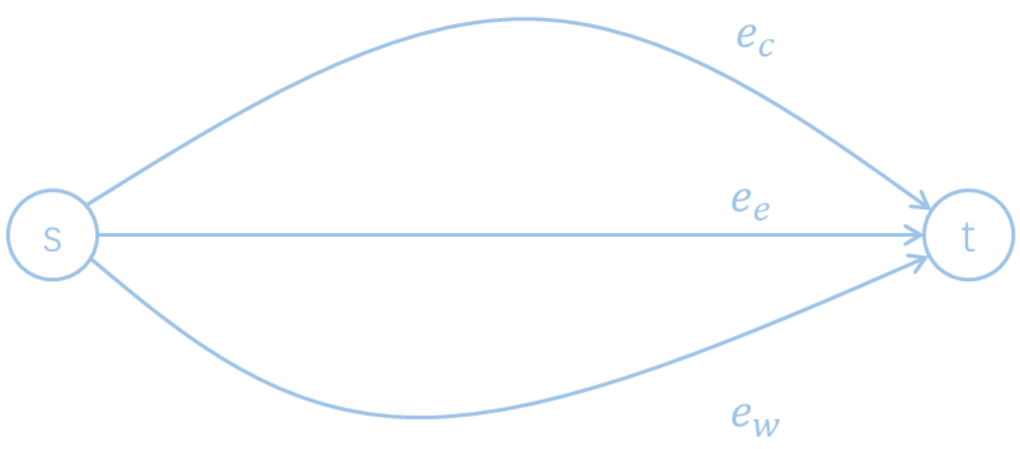

Figure 2: model of the tunnel system

The tunnel system is modeled as a directed graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ with the starting point $\boldsymbol{s}$ and the ending point $\boldsymbol{t}$ (see Figure 1). The three directed edges $(\boldsymbol{e_C},\ \boldsymbol{e_E},\ \boldsymbol{e_W})$ in the figure represent the three tunnels (CHT, EHT, WHT) respectively. The $\boldsymbol{s}$ and $\boldsymbol{t}$ represent the Kowloon and the Hong Kong Island. We use the vertex $[\boldsymbol{c_C}, \boldsymbol{c_E}, \boldsymbol{c_W}]^{\boldsymbol{T}}$ to represent the holding capacity of the three tunnels, which is the largest number of cars the tunnels can hold without queues.

We define the amount of traffic using each tunnel as *a flow on each edge*, denoted by, $\boldsymbol{f_e}$. Then, we use a positive, non-decreasing, and continuous latency function $\boldsymbol{l_e}$ describing the delay caused by the flow of the edge. Thus, the total delay of the graph is

$$\mathbf{L}(\mathbf{G}) = \sum_{\boldsymbol{e} \in \boldsymbol{E}} (\boldsymbol{l_e}(\boldsymbol{f_e}) * \boldsymbol{f_e}).$$

For simplicity, we could define $\boldsymbol{l_e}(\boldsymbol{f_e}) = \boldsymbol{f_e}/\boldsymbol{c_e}$.

The minimal-latency is represent as $\mathbf{L}(\cdot)$. [8]

Besides that, we represent the toll rate as $\boldsymbol{t_e}$ for each tunnel. The toll rates can either be fixed or be a function changing with the flow on the edge.

The tunnel system is modeled as a directed graph because the traffic from Kowloon to Hong Kong Island is independent of the traffic from Hong Kong Island to the Kowloon. As a result, investigation into only one direction of the tunnels is enough. Meanwhile, the traffic capacity of the three tunnels are evenly distributed between the two directions. Besides that, the traffic starts form Kowloon to Hong Kong Island in the morning returns to Kowloon at night. Therefore, we can reveal the full information of the tunnel system by evaluating the model twice with different parameters of the two directions.

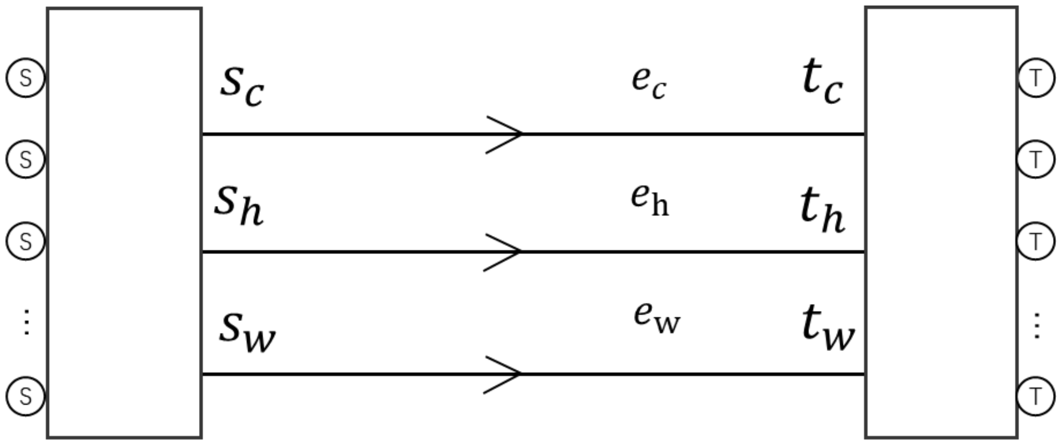

Figure 3: intuitive model of the tunnel system

That the three edges share the same starting point and ending point is also reasonable. Intuitively, the model of the tunnel system should be Figure 3. The circles with S represent the starting points of the drivers and the circles with T represents the destinations of the drivers. They will choose any of the three tunnels through the roads covered by the rectangle. This model seems to be more reasonable than the previous one. However, large numbers of drivers use the three tunnels every day. Therefore, a single pair of starting point and ending point does not affect the whole system that much. Meanwhile, the condition of roads connecting these locations and the three tunnels does not change greatly with the change of allocation of traffic over the tunnels. In other words, the latency of these roads covered by the rectangles does not change when the flows of the tunnels change. Therefore, the starting points, the destinations and the ways to the three tunnels can all be viewed as the two points ***s*** and ***t*** in the Figure 2.

3.1.2. Model of Drivers and Cost Function

As mentioned before, we assume that all drivers are selfish based on actual experience. In other words, they care only about their own benefits and care nothing about the whole performance of the tunnel system. Meanwhile, the drivers mainly take the toll rates and the travel time into consideration when choosing the tunnel. Particularly, all drivers prefer shorter latency to longer latency as well as prefer lower toll rate to higher toll rate.

It is reasonable to assume that the drivers are selfish game players because the drivers using the tunnels can hardly communicate with each other and the cooperation between large number of people without an authority is always impossible. Meanwhile, we cannot expect there are always some people willing to sacrifice their own benefits for a better overall performance. Therefore, that the drivers care only about time and money makes sense.

Each driver has their own evaluation of the importance of the latency and the toll rates. For a driver $\boldsymbol{d}$, an edge's cost equals to:

$$\boldsymbol{C_d = a_d(l_e(f_e)) + b_d(t_e(f_e)),}$$

where the positive, non-decreasing, and continuous functions $\boldsymbol{a}$ and $\boldsymbol{b}$ indicate the importance of the two factors for the users respectively [8]. A driver prefers choosing the edge with the lowest cost.

The network reaches a Nash equilibrium when for each user and edge $e \in$ E:

$$\boldsymbol{a_d(l_{e^*}(f_{e^*})) + b_d(t_{e^*}(f_{e^*})) \leq a_d(l_e(f_e)) + b_d(t_e(f_e))},$$

where e^* is the edge the user chooses. Since such a Nash equilibrium exists in every such network according to theorem 2.2.1., investigating the model is meaningful and practical.

3.2. Scenario Based on the Real Situation

With theoretical model constructed, we can now apply the model to the real situation and build a scenario based on the actual situation of the Hong Kong tunnel system. we will assign parameters to the functions we previously mentioned and further enrich the possible behaviors of the road users in that scenario.

We set a scenario where the three tunnels can hold 100, 200 and 300 cars every time, denoted as: $\begin{cases} \boldsymbol{c_C = 100} \\ \boldsymbol{c_E = 200} \\ \boldsymbol{c_W = 300} \end{cases}$. The different capacities are set for better fitting the real situation of the three tunnels.

When the tunnel is not congested, the time required for one car go through one tunnel is same for the three tunnels.

We set 1200 cars to go form Kowloon to Hong Kong Island every day, which exceeds the holding capacity of the whole system to simulate the real congestion.

Combined with the model mentioned in section 3.1.1, we have: $\begin{cases} \boldsymbol{l_C = \frac{f_C}{c_C} = f_C/100} \\ \boldsymbol{l_E = \frac{f_E}{c_E} = f_E/200} \\ \boldsymbol{l_W = \frac{f_W}{c_W} = f_W/300} \end{cases}$.

For the user part, we assume that group characteristics rather than personal characteristics are observed and all drivers share the same cost function. For simple computation, the cost function of the driver can be expressed as:

$$\boldsymbol{C = a * l_e(f_e) + b * t_e(f_e)},$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constants. Even though this function looks simple, it reflects the basic characteristics of the drivers. A driver will choose a tunnel with shorter travel time and lower toll rate. Therefore, the function is reasonable. In this scenario, we set $\begin{cases} \boldsymbol{a = 1} \\ \boldsymbol{b = 0.1} \end{cases}$. It should be noticed that the value of $\boldsymbol{a}$ and $\boldsymbol{b}$ are not important because $\boldsymbol{a}$ and $\boldsymbol{b}$ vary with the way of expressing latency and toll rates. $\boldsymbol{a}$ is set as 1 for simpler calculation.

Finally, we set a simulation of one hundred days. Each user will take the congestion situation of the tunnel system yesterday and the toll rates of the three tunnels today into consideration when choosing which tunnel to use. The user ranks the three tunnels according to his cost function and prefers the tunnel with lowest cost.

It should be admitted that the scenario is a hypothetical one and the parameters used are mainly based on theoretical assumption. However, since the scenario reveals the main characteristics of the congestion problem, it can be used as a starting point for handling the real problem. Besides that, the model offered is in fact universal, so further research can be conducted by adjusting the parameters in the model for better fitting the real situation.

3.3. Procedure of Analyzing System Performance

After constructing the model, we turn to how to analyze the result computed by the model.

In the previous section, we construct a scenario where the three tunnels can hold 100, 200 and 300 cars separately every time. The whole system serves 1200 cars every day. The user chooses the tunnel with lowest cost. We aim at minimizing the total time spent by the users. Mathematically, the whole problem can be described as:

$$\mathbf{min\,L(G)},$$

where:

$$\boldsymbol{l_C = f_C / c_C = f_C / 100}$$

$$\boldsymbol{l_E = f_E / c_E = f_E / 200}$$

$$\boldsymbol{l_W = f_W / c_W = f_W / 300}$$

$$\boldsymbol{f_C + f_E + f_W = 1200}$$

$$\mathbf{L(G)} = \boldsymbol{l_C * f_C + l_E * f_E + l_W * f_W}$$

The $[\boldsymbol{f_C}, \boldsymbol{f_E}, \boldsymbol{f_W}]^T$ is made up of the aggregative behavior of the road users who act according to the cost function:

$$\boldsymbol{C = l_e(f_e) + 0.1 t_e(f_e)}.$$

We can easily find that the optimal latency is achieved when $\begin{cases} \boldsymbol{f_C = 200} \\ \boldsymbol{f_E = 400} \\ \boldsymbol{f_W = 600} \end{cases}$,

with $\mathbf{L(G) = 2400}$ with the help of Cauchy–Schwarz inequality.

According to according to theorem 2.2.1, we can compute the equilibrium in face of different toll rates. The equilibrium situation can be attained by help of programs. After that, we use POA to judge whether the system performs well or not. Let $\mathbf{L(G)}$ and $\mathbf{L(\cdot)}$ denote equilibrium and optimal latency respectively. We have:

$$\mathbf{POA} = \frac{\mathbf{L(G)}}{\mathbf{L(\cdot)}} = \frac{\mathbf{L(G)}}{\mathbf{2400}}.$$

The system's performance is better when POA is closer to 1.

4. Project Result Analysis

Since the project aims at re-distributing the traffic flow in the tunnel system by adjusting toll rates, the toll rates are viewed as a set of variables while other parameters are viewed as constants in the following part of report. This project mainly investigates three different models. First of all, we build a model without toll rates. Then, we build a model with fixed toll rates. Finally, the model with fluctuated toll rates is looked into.

In each model, we start from different initial distribution of traffic, and figures out whether the system reaches an equilibrium. After getting the equilibrium, we judge the performance by calculating the POA of the equilibrium situation.

In the following part, we use $(\boldsymbol{f_C}, \boldsymbol{f_E}, \boldsymbol{f_W})$ to represent the flow distribution and $[\boldsymbol{t_C}, \boldsymbol{t_E}, \boldsymbol{t_W}]$ to represent the toll rates. For example, (200, 300, 700) means that 200 cars use the CHT, 300 cars use the EHC and 700 cars use the WHC. Similarly, the [20, 30, 40] means that the CHT charges twenty dollars, the EHC charges thirty dollars and the WHC charges forty dollars.

4.1. Scenario without toll rates

To start with, we first look at the situation where no toll rates are charged. The cost function here is $\boldsymbol{C} = \boldsymbol{l_e}(\boldsymbol{f_e})$ since the toll rates equal to one.

Initially, we assume that all drivers are fully rational, that is, they choose the tunnel according to cost function strictly. When two tunnels have the same cost, they choose either one of the tunnels.

Day	0	1	2	3	4	5	6	7	…
CHT	400	0	600	0	600	0	600	0	…
EHC	400	0	600	0	600	0	600	0	…
WHC	400	1200	0	1200	0	1200	0	1200	…

Table 1.1: the every day flow of each tunnel with initial state (400, 400, 400)

Day	0	1	2	3	4	5	6	7	…
CHT	N/A	4	0	6	0	6	0	6	…
EHC	N/A	2	0	3	0	3	0	6	…
WHC	N/A	1.7	3	0	3	0	3	0	…

Table 1.2: the cost of each tunnel at the beginning of everyday with initial state (400, 400, 400)

However, from table 1.1. and 1.2, we find that the traffic flow keeps fluctuating among the three tunnels. This is because if the three tunnels have different cost at the initial state, all the drivers shift to the tunnel with least cost on the first day. Then on the second day, the drivers choose the first day’s two empty tunnels. On the third day, all the drivers choose the one empty tunnel not chosen by anyone on the second day, which is the same as the first day. As a result, the system loops with two day as a unit infinitely.

To avoid this situation, we adjust the behavior of the users. Instead of following the cost function strictly, the users now just use the cost function as a reference. When a driver chooses which tunnel to use, he ranks the three tunnels according to the cost function. If the tunnel he chose yesterday is the worst road today, he has 2% possibility to choose the best tunnel, 1% possibility to choose the second-best tunnel and 97% possibility to stay the same. If the road he chose yesterday is the second-best tunnel today, he has 1% possibility to choose the best tunnel and 99% possibility to stay the same. If the tunnel he chose yesterday is the best one today, he chooses the same tunnel. The possibility is set based on the fact that worse tunnel one uses, the stronger incentive one has to make a change and the fact that the tunnel with lower cost is more attractive.

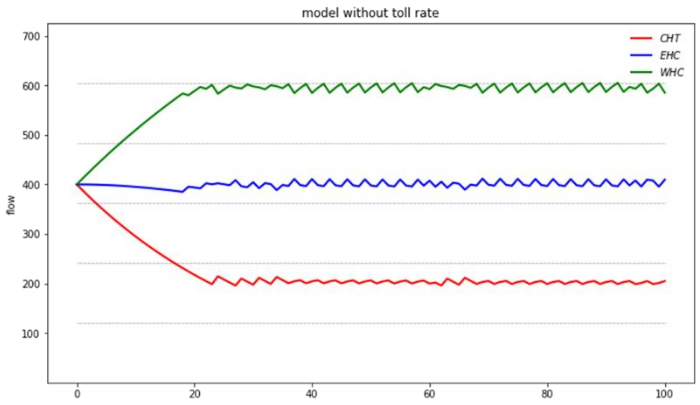

Figure 4.1: the flow of each tunnel with initial state (400, 400, 400)

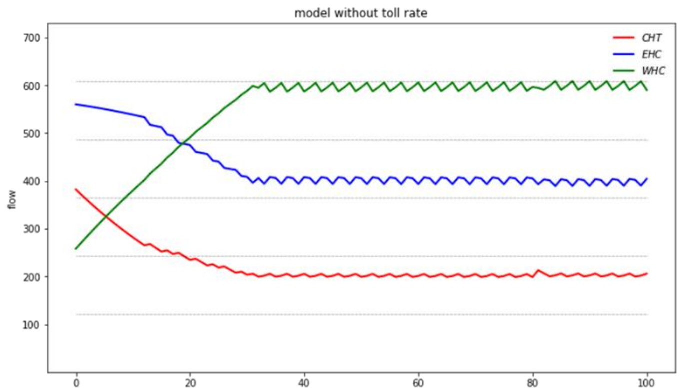

Figure 4.2: the flow of each tunnel with initial state (382, 560, 258)

With the adjusted user actions, we get the line chart as the figure 4.1., 4.2. and 4.3. shows. According to figure 4.1 and figure 4.2, the distribution of the road system converges to (200, 400, 600) regardless of initial situation, which is the same with the optimal case. The result coordinates with the Wardrop Equilibria in that it both reaches the user equilibrium and the system optimal.

4.2. Scenario with fixed toll rates

The scenario with fixed toll rates is investigated for two reasons. On the one hand, we want to prove that different toll rates result in different equilibria, indicating that toll rates are useful for regulating traffic. On the other hand, we want to prove that it is possible to optimize the traffic flow by properly setting toll rates.

4.2.1. Scenario with the Same Fixed Toll Rates and Different Initial States

Before looking into how toll rates influence the performance of the system, we need to first find out whether the initial state has effect on the equilibrium state of the system or not.

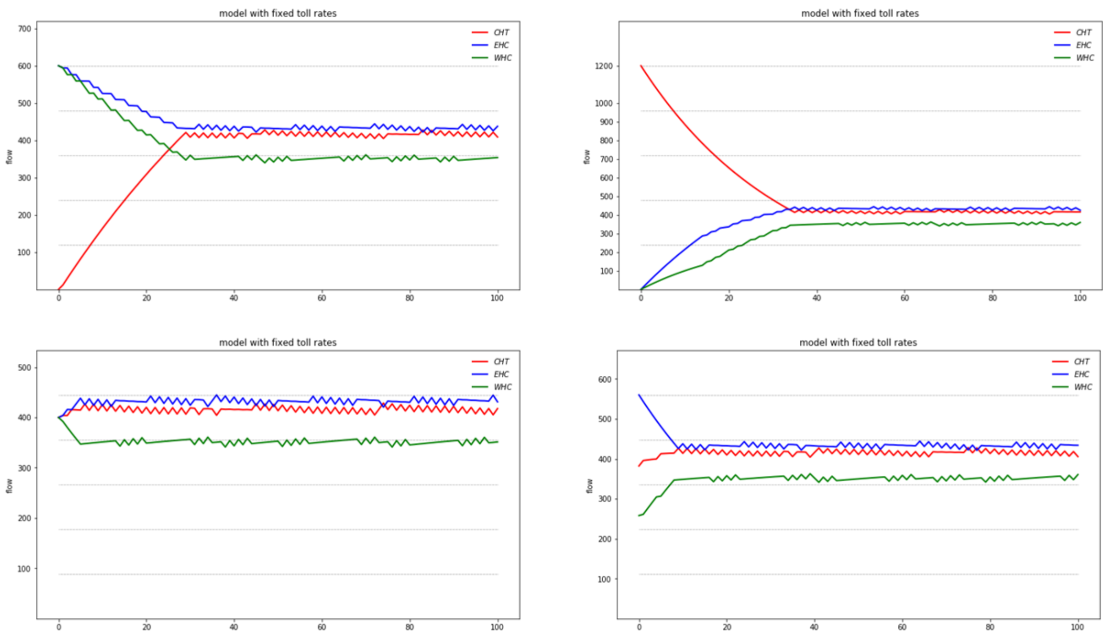

Figure 5.2: the flow of each tunnel with initial state [20, 40, 50]

As figure 5 shows, we compute four randomly selected different initial states with the same toll rates [20, 40, 50]. They all reach the same equilibrium. The system stays stable after day 40.

In fact, that the initial state has effect on the equilibrium state of the system can be proved mathematically. As discussed in the previous session, the system reaches equilibrium when no one could benefit by changing his or her own strategy. In other words, the user's cost functions in the equilibrium system have the same value. Use the toll rates in figure 5 as an example, we have

$$\begin{cases} \boldsymbol{C = l_C(f_C) + 0.1t_C(f_C) = \frac{f_C}{100} + 2} \\ \boldsymbol{C = l_E(f_E) + 0.1t_E(f_E) = \frac{f_E}{200} + 4} \\ \boldsymbol{C = l_W(f_W) + 0.1t_W(f_W) = \frac{f_W}{300} + 5} \\ \boldsymbol{f_C + f_E + f_W = 1200} \end{cases}$$

Solving the above equation, we have $\boldsymbol{f_C = 417, f_E = 433, f_W = 350,}$ which coordinates with figure 5.

4.2.2. Scenario with Different Fixed Toll Rates

Before turning to program for help, we may first analyze the problem with mathematics. Just as mentioned in the previous part, the equilibrium situation can be described as:

$$\begin{cases} C = l_C(f_C) + 0.1t_C(f_C) = \dfrac{f_C}{100} + 0.1t_C \\ C = l_E(f_E) + 0.1t_E(f_E) = \dfrac{f_E}{200} + 0.1t_E \\ C = l_W(f_W) + 0.1t_W(f_W) = \dfrac{f_W}{300} + 0.1t_W \\ f_C + f_E + f_W = 1200 \end{cases}$$

Solving the above equation, we get

$$\begin{cases} f_C = 200 - \frac{25}{3}t_C + \frac{10}{3}t_E + 5t_W \\ f_E = 400 + \frac{10}{3}t_C - \frac{40}{3}t_E + 10t_W \\ f_W = 600 + 5t_C + 10t_E - 15t_W \end{cases}.$$

From the set of equation above, we can conclude that different set of toll rates will result in different equilibrium.

Besides that, since f_C, f_E and f_W should all be larger than or equal to 0, we have:

$$\begin{cases} f_C = 200 - \frac{25}{3}t_C + \frac{10}{3}t_E + 5t_W \geq 0 \\ f_E = 400 + \frac{10}{3}t_C - \frac{40}{3}t_E + 10t_W \geq 0 \\ f_W = 600 + 5t_C + 10t_E - 15t_W \geq 0 \end{cases}.$$

That is:

$$\begin{cases} t_C - t_E + 120 \geq 0 \\ t_E - t_C + 120 \geq 0 \\ t_W - t_C + 40 \geq 0 \end{cases}, (1)$$

which indicates that we should not break the inequality group (1) if we want to use fixed toll rates to adjust the flow of the system.

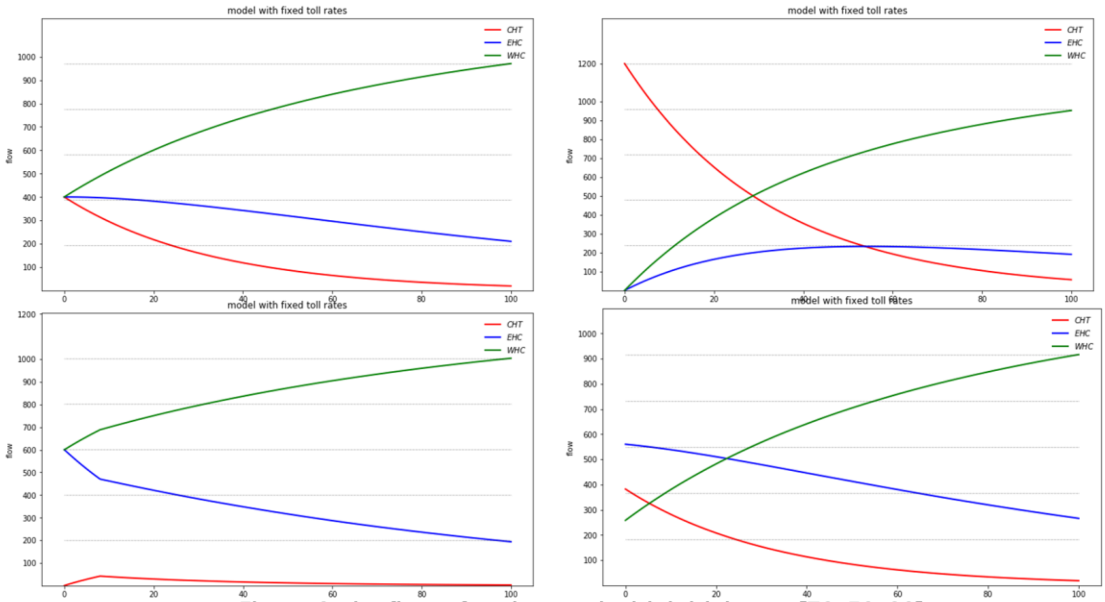

Figure 6: the flow of each tunnel with initial state [70, 50, 20]

From figure 6, we can observe that the system will not converge if the inequality group 1 is violated. This is because when the rule is violated, the toll rate element denominates the cost function and no matter how latency varies, the users' choice will not change as long as they prefer lower cost to the higher cost.

4.2.3. Exploration of Optimal Situation of the Scenario with Fixed Toll rates

According to the discussion of section 3.3, the system reaches both optimal and equilibrium when $\begin{cases} f_C = 200 \\ f_E = 400 \\ f_W = 600 \end{cases}$. Combined with the equalities used in the last session, we get $\boldsymbol{t_C = t_E = t_W}$. However, when the toll rates of the three tunnels are the same, the toll rates become meaningless in the field of adjusting travel flows. Therefore, we can conclude that we cannot reach the optimal situation with fixed toll rates. However, this does not mean that toll rates are meaningless in reality since controlling the travel flow is only part of their functions.

4.3. Scenario with Dynamic Toll Rates Changing with the Latency

As we have proven that the optimal situation cannot be achieved by fixing toll rates, dynamic toll rates seem to be a wiser choice because the system can react to the traffic situation in a flexible way.

To mitigate the congestion, we should raise the price of the congested tunnel and lower the price of the relatively spare tunnel. To model this kind of change, the toll rate of the tunnel with highest latency yesterday will be increased by ten percent and the toll rate of the tunnel with lowest latency yesterday will be lowered by ten percent.

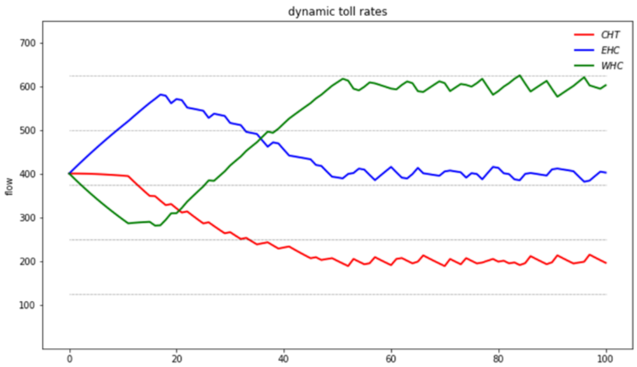

Figure 7.1: the flow of each tunnel with initial state (400, 400, 400) and [20, 40, 50]

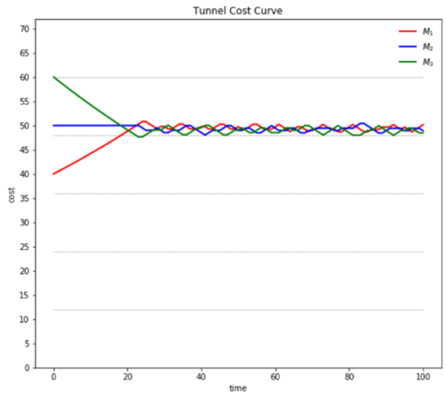

Figure 7.2: the toll rates of each tunnel with initial state (400, 400, 400) and [20, 40, 50]

The whole system gradually reaches an equilibrium (see Figure 7.1) with the toll rates of the three tunnels being the same (see Figure 7.2), indicating that the system becomes both stable and optimal in this case according to section 4.2.3. Even though the dynamic toll rates model seems to reach the same result of directly fixing the three toll rates, the dynamic way outperforms the fixed way in many two folds.

On the one hand, the price at equilibrium is calculated by algorithm rather than the authority. It is usually very difficult for one government to set a very accurate price to maximize the benefit of the whole society. By offering the initial toll rates, a rough bound is set and the efficiency for deciding the toll rates are saved.

On the other hand, the dynamic price method can correct the pricing

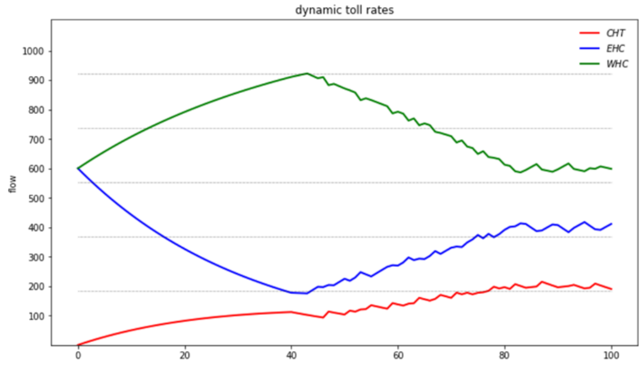

Figure 8.1: the flow of each tunnel with initial state (600, 600, 0) and [70, 50, 20]

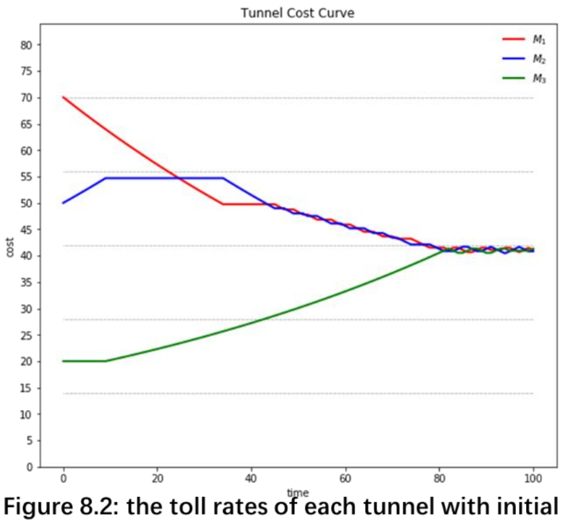

Figure 8.2: the toll rates of each tunnel with initial state (600, 600, 0) and [70, 50, 20]

error by itself. For example, if we use an initial state violating the inequality group (1), the dynamic price method can gradually changing the toll rates and reduce potential waste. As shown in Figure 8.1, the system changes from divergence to convergence after the day 40. Figure 8.2. shows that the mistaken price is corrected gradually every day.

5. Limitations and Future Work

It should be admitted that the model now is a conceptual one. Even though this project tries to comply with the basic characteristics of the whole system, some computation methods might be too simple to completely reveal the whole situation. For example, the cost function is identical for every user in the mentioned model, while in reality, different people might have different view of the time and money.

As for the dynamic toll rate part, this project only proves that the system can reach a stable state with the help of toll rates. However, how to change the toll rates in a rational way now is lack of theoretical support. What we have done now is based more on intuition rather than mathematics.

In the future work, more data should be collected to build a more precise cost function. Meanwhile, how to dynamically adjust the toll rates for better system performance should be looked into.

6. Conclusion

This project draws a brief picture of the Hong Kong cross-harbor system and seeks how to alleviate or solve the congestion problem of the tunnels by changing the toll rates. By modeling the system as a congestion game, the project shows that toll rate is a useful tool in redirecting part of the traffic of one tunnel to other tunnels.

The project shows that a congested system can reach optimal case by itself when toll rates are in absence. Besides that, the toll rates do have great influence on the distribution of flow. Systems with different toll rates usually reach different equilibrium. By setting proper set of toll rates, the whole system could reach an equilibrium where the whole performance of the system is optimal. As for the dynamic toll rates part, the project proves that the system can reach equilibrium when the system's sensitivity to toll rates is high or in other words, when the adjustment of the toll rates is high.

Future work can be focused on building a more realistic model. On the one hand, the cost functions may vary for heterogeneous users, making the modeling of driver more complicated. On the other hand, more effort should be devoted to building a both stable and optimal system with dynamic pricing strategy.

7. Conclusion

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