On Hamiltonian Tetrahedralizations Of Convex Polyhedra

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Abstract. Let T_p denote any tetrahedralization of a convex polyhedron P and let G^T be the dual graph of T_p such that each node of G^T corresponds to a tetrahedron of T_p and two nodes are connected by an edge in G^T if and only if the two corresponding tetrahedra share a common facet in T_p. T_p is called a *Hamiltonian tetrahedralization* if G^T contains a Hamiltonian path (HP). A well-known open problem in computational geometry is: can every polytope in 3D be partitioned into tetrahedra such that the dual graph has an HP? In this note, we shall show that there exists a 92-vertex polyhedron in which the pulling method does not yield a Hamiltonian tetrahedralization, here the pulling method is the simplest method to ensure a linear-size decomposition and is one of the most commonly used tetrahedralization methods for convex polyhedra. Furthermore, we can construct a convex polyhedron with n vertices such that the longest path in the dual graph in question can be as short as $O(1)$. This fact suggests that it may not be possible to find a good approximation of a HP for convex polyhedra using the pulling method.

1 Introduction

A tetrahedralization of a convex polyhedron P, denoted by T_p, is a partition of P by a set of simplices called tetrahedra. Tetrahedralization T_p has a number of interesting properties. For example, the number of tetrahedra in different tetrahedralizations of a polyhedron with n vertices may vary from $\theta(n)$ in the low end to $O(n^2)$ in the high end. The most commonly used tetrahedralization methods are *pulling* and *shelling* (the latter includes *plane-sweeping* as a special case). In the pulling method, a vertex v of P, called an *apex*, is connected to all other vertices of P by edges to form a tetrahedralization of P, and in the shelling method, a *cap* is removed from P and tetrahedralized at each step, where a cap is a space between the convex hulls $CH(P)$ and $CH(P-\{v\})$ for a vertex v of P. Note that pulling ensures a linear number of tetrahedra while shelling may generate $\Theta(n \log n)$ tetrahedra in the worst case [3]. Furthermore, finding a tetrahedralization of P with the minimum number of tetrahedra, called an *optimal tetrahedralization*, has been proved to be NP-Complete [2], and the best approximation ratio one can obtain is $2 - \frac{1}{\sqrt{n}}$ [4]. This fundamental geometry structure also has many applications. For example, in computer graphics, the performance of certain rendering algorithms is closely related to the quality of tetrahedralization, and in progressive transfer of figures in computer network, the performance is closely related to the data rate. In particular, one hopes that the dual graph of the tetrahedralization contains an HP so that the data rate to the algorithms or to the networks can be reduced. While every 2D polygon has a triangulation with an HP, it is not known if this holds even for 3D convex polyhedron. It is conjectured that there always exists a Hamiltonian tetrahedralization for any convex polyhedron [1], and the associated problem later has been listed in the open problem project [5]. In this note, we show that there exists a convex polyhedron Q in which pulling does not yield a Hamiltonian tetrahedralization. The ultimate solution to this problem is still elusive. We shall also show that no tetrahedralization obtained by pulling admits a good approximation to an HP, in terms of a fraction of n vertices. As a minor result, we present a 94-node 3-regular and 3-connected planar graph without HP.

2 A convex polyhedron in which pulling does not yield a Hamiltonian tetrahedralization

We first describe a basic convex polyhedron building block and its properties. We then construct a convex polyhedron using these blocks and show that the resulting polyhedron does not have a Hamiltonian tetrahedralization by pulling. This building block uses the so-called Tuttes non-HC graph component [6].

2.1 The 2D basic case

Let us consider a triangulation of 10 vertices in 2D and its dual graph. This dual graph of 25 nodes is a component of a 3-regular and 3-connected graph which does not have Hamiltonian circle [6]. Observe that the dual graph $G^{T_{10}}$ has exactly three degree-2 nodes and there is no HP for certain pair of start and end nodes, in particular, degree-2 nodes a and c (refer to Figure 1).

We use this graph to build a larger graph of 45 nodes, in which no Hamiltonian paths exist. The corresponding triangulation has 25 vertices in the plane (refer to the lefthand side of Figure 2). It will be verified in Lemma 2 that any planar graph G containing this 45-nodes subgraph would have to leave at least one end of an HP in this subgraph if G has any HP (refer to the righthand side of Figure 2).

Lemma 1. *There is no HP in* $G^{T_{25}}$.

Proof. Note that any path which has traversed the first $G^{T_{10}}$ and entered the second $G^{T_{10}}$ must start at node a (respectively, node c) of the second $G^{T_{10}}$.

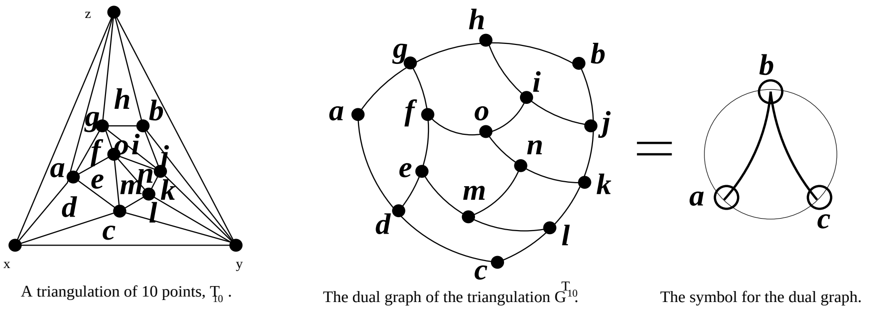

Fig. 1. A dual graph without Hamiltonian path of fixed-end between nodes a and c.

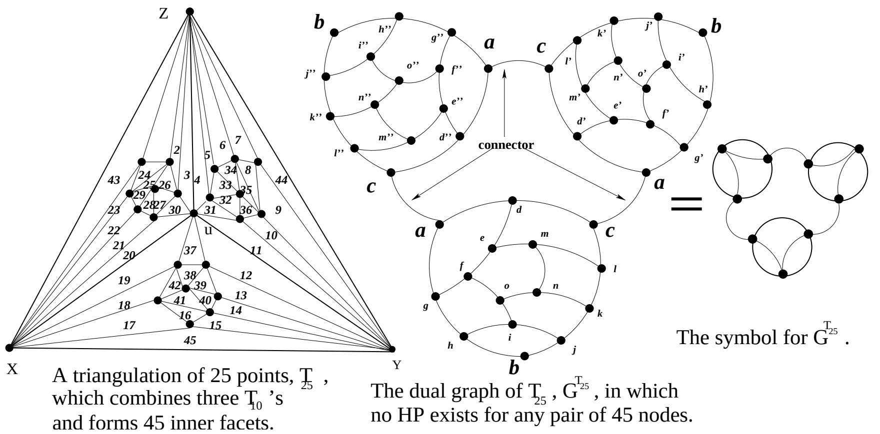

Fig. 2. A dual graph without Hamiltonian path between any pair of its 45 nodes.

Hence, the path cannot traverse the second $G^{T_{10}}$ with node c (respectively, node a) as the last node by our previous observation. Consequently, the longest path L in $G^{T_{25}}$ can cover at most two $G^{T_{10}}$'s, and L is not an HP in $G^{T_{25}}$.

Lemma 2. *Any graph G containing $G^{T_{25}}$ as subgraph and connecting $G - G^{T_{25}}$ to $G^{T_{25}}$ through three nodes of degree 2 (node b) has to leave at least one end of an HP in $G^{T_{25}}$ if G has such an HP.*

Proof. Let L be a longest path in G. If L starts inside $G^{T_{25}}$, the lemma is done. Otherwise, by the given connection restriction between $G - G^{T_{25}}$ and $G^{T_{25}}$ and by Lemma 1, the longest segment of L inside $G^{T_{25}}$ can only cover two of the $G^{T_{10}}$ of $G^{T_{25}}$. Then, the rest of L must return to $G - G^{T_{25}}$ and re-enters $G^{T_{25}}$ through the connector of the third $G^{T_{10}}$. Then L must terminate inside $G^{T_{25}}$ since two other connectors are already used.

Now we combine three T_{25}'s to form a new triangulation T_{70} (refer to Figure 3). We shall show that its dual graph $G^{T_{70}}$ does not have an HP and any graph G containing $G^{T_{70}}$ would not have an HP either.

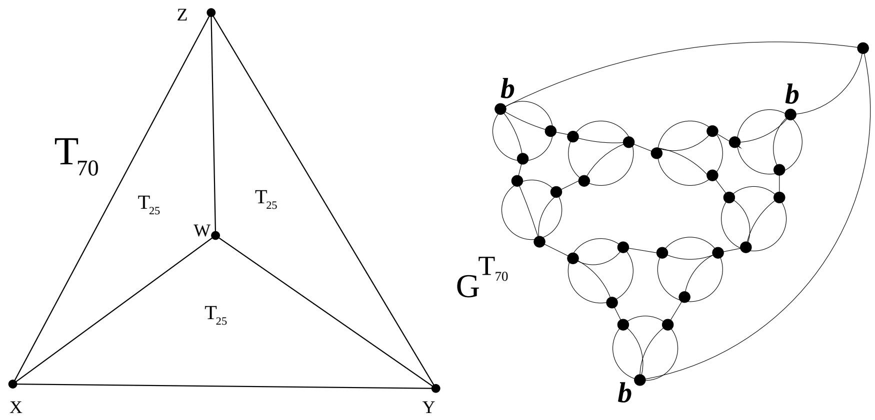

Fig. 3. Dual graph $G^{T_{70}}$ does not have a Hamiltonian path.

Lemma 3. *There is no HP in $G^{T_{70}}$. Moreover, no graph G containing $G^{T_{70}}$ as a subgraph and connecting $G - G^{T_{25}}$ and $G^{T_{25}}$ through three degree-2 nodes in $G^{T_{70}}$ has an HP.*

Proof. Let L be a longest path in G. By Lemma 2, the longest segment of L can cover at most two $G^{T_{10}}$'s in a $G^{T_{25}}$. Path L must either leave $G^{T_{70}}$ or enter one of the two neighboring $G^{T_{25}}$'s through a connector. Thus, if L does not leave $G^{T_{70}}$, the longest segment of L can cover at most seven $G^{T_{10}}$'s and leave one end of L inside $G^{T_{70}}$ so that $G^{T_{70}}$ also has two uncovered disjoint $G^{T_{10}}$'s. Note that when L leaves $G^{T_{70}}$ and reenters $G^{T_{70}}$, it can only cover one of these two disjoint $G^{T_{10}}$'s. Thus, eventhough L covers all of $G - G^{T_{70}}$, it is still not an HP in G.

2.2 Constructing the polyhedron

In this section, we describe how to construct a convex polyhedron Q without a Hamiltonian tetrahedralization relative to the pulling method using the 2D units observed in Section 2.1.

Let abc, abe, bcf, and cag be four neighboring triangles on the surface of a convex polyhedron. The *bounding tetrahedron* associated with triangle abc, say $abcd$, is one in which the dihedral between abc and abd equals the dihedral between abe and the plane extending abd, and a similar equality holds for the dihedrals related to bcd and for the dihedrals related to cad, respectively. (Refer to part (a) of Figure 4.)

Roughly speaking, we first build a regular tetrahedron. For each of the four triangular faces of the tetrahedron, we patch a basic building block inside its corresponding bounding tetrahedron, where the block is a 3D convex hull of 25 vertices with triangular face (refer to part (b) of Figure 5). Each basic building block contains three similar basic parts (sharing some boundaries), and each basic part contains a convex hull of 10 vertices inside its corresponding bounding

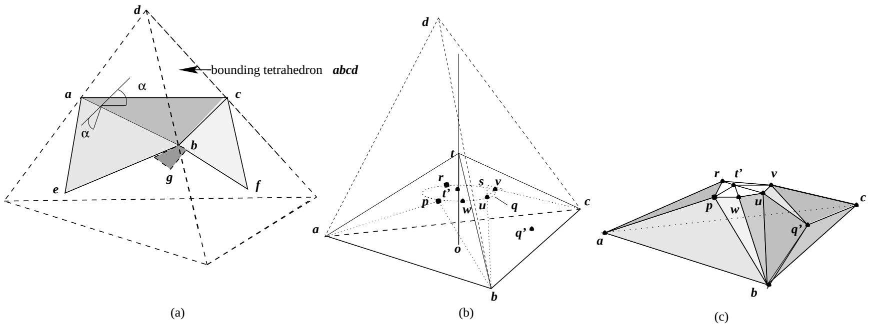

Fig. 4. The surface graph of the constructed convex hull of 10 points is homeomorphic to T_{10}. The vertices of the two graphs have the following correspondence: $(a \longleftrightarrow 3)(b \longleftrightarrow 2)$ $(c \longleftrightarrow 1)(v \longleftrightarrow 4)$ $(q' \longleftrightarrow 5)(u \longleftrightarrow 6)$ $(w \longleftrightarrow 7)(p \longleftrightarrow 8)$ $(r \longleftrightarrow 9)(t' \longleftrightarrow 10)$
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tetrahedron. The graph of this convex hull is homeomorphic to that of the desired triangulation (refer to part (a) of Figure 5).

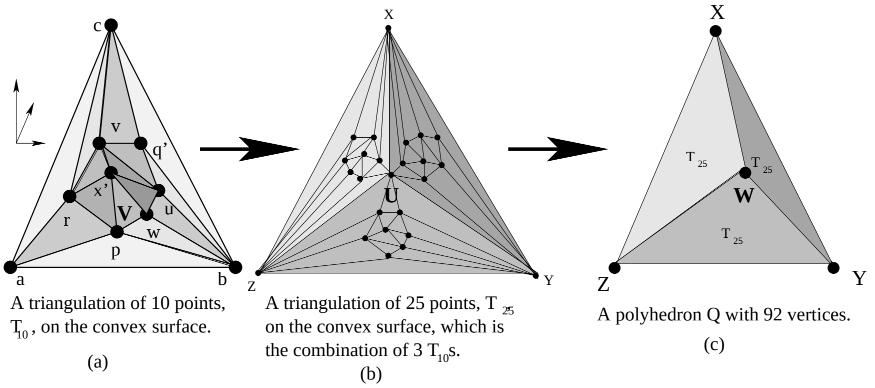


Fig. 5. The surface graph of the convex hull of 10 points is homeomorphic to that of T_{10}; Three T_{10} surfaces with some overlapped boundary edges and vertices form a surface of a basic building block. Four such blocks patched on a regular tetrahedron form a Q.

In more detailed constructions of a 10-vertex basic part, one may refer to Figure 4. Let $abcd$ be a bounding tetrahedron. Let o be the common point of the three inner-angle-bisectors of triangle abc. Let t be the midpoint of the line segment which is vertical to abc, lies inside $abcd$, and ends at o. Let circle C, be centered at the midpoint of line segment ot, be vertical to ot and be touching the boundary of tetrahedron $abct$. Let p, q, and r be the respective tangent points of C from the plane containing edge ab, the plane containing bc, and the plane containing ca. Let u and v be the intersection points between the arc pq and plane containing $bcuv$ and between the arc qr and this plane, respectively. Note that u and v are very close to q. Let t' be a point on the line segment connecting t and the center of C. Let q' be a point close to face $bcuv$. The points t' and q' is so placed that line segment $t'q'$ lies right below line segment uv. We now place vertices on points $a, b, c, p, q', r, u, v, w$ and t' as described above. It can be verified that the convex hull of these ten vertices, excluding triangle abc, forms a surface graph isomorphic to T_{10}. Let CH_{abc} denote this convex hull (refer to part (c) of Figure 4). To build a basic building block, for a triangular face of the regular tetrahedron, say XYZ, we place a vertex U on the middle point of the line segment (constructed similarly to ox in the basic part) inside the bounding tetrahedron of XYZ. Then, for each of trianglar faces XYU, YZU, and ZXU, we attach a basic part inside the corresponding bounding tetrahedron. The convex hull of the 25-vertex building block forms a surface graph homeomorphic to the triangulation T_{25} as shown in the lefthand side of Figure 2. Polyhedron Q is formed by attaching each of the four triangular faces of a regular tetrahedron with a basic building block inside the corresponding bounding tetrahedron.

The above construction has two properties:

Property 1: *Let CH_{abc} be a basic part patched on a triangular face abc of a convex polyhedron Q. Then, the new polyhedron $Q' = Q \cup CH_{abc}$ is convex.*

This is because the basic part is placed in its corresponding bounding tetrahedron with respect to Q. The restriction of bounding tetrahedron ensures that the convexity of the neighboring pieces in the surface of Q' does not be violated.

Let Q be a 'general' convex polyhedron, i.e., all the faces of Q are planar triangles.

Property 2: *The new polyhedron $Q' = Q \cup CH_{abc}$ is also a general convex polyhedron.*

This is because the space between the bounding tetrahedron $abcd$ and CH_{abc} is always non-empty due to our construction. Hence, the subquent faces are all triangles and their corresponding bounding tetrahedra are non-empty too.

Property 2 implies that the above convex hull patch process can be recursively executed, and the number of vertices of Q' can be increased to as large as necessary.

2.3 The triangulated surfaces viewing from different pulling apecies

In this section, we shall describe the internal facet-graph of such a convex polyhedron Q as viewed from a pulling apex. The corresponding dual graph is also presented.

Let W be a shared vertex of three T_{25}'s in Q, and let U be a shared vertex of the three T_{10}'s, and let V be one of the remaining vertices in T_{10} (refer to the bottom part of Figure 5). We now consider three cases and subcases according to these three types of apecies.

1. Case 1 shows the facet graph of Q viewed from apex W. The dual graph G contains a subgraph $G^{T_{25}}$ (refer to Figure 6).
2. Case 2(a) shows the facet graph of Q viewed from apex U. The dual graph G contains a subgraph $G^{T_{70}}$ and $G - G^{T_{70}}$ contains three disjoint components (refer to Figure 7).
3. Case 2(b) shows the facet graph of Q viewed from apex V. The dual graph G contains a subgraph $G^{T_{70}}$ and $G - G^{T_{70}}$ is a connected component (refer to Figure 8).

Theorem 1. *There exists a convex polyhedron Q with 92 vertices in which pulling does not yield a Hamiltonian tetrahedralization.*

Proof. The vertices of Q can be classified into three groups depending on what type of subgraphs Q's dual graph G^{T_Q} contains. In Case 1, G^{T_Q} contains a $G^{T_{25}}$ as well as nine bridge edges. Each component of $G^{T_Q} - G^{T_{25}}$ which connected to $G^{T_{25}}$ must contain one end of a path since they are disjoint. There are three such components. Therefore, the longest path cannot be an HP in G^{T_Q}. In Case 2 (a) and (b), G^{T_Q} contains $G^{T_{70}}$ as a subgraph, and by Lemma 3, there is no HP in G^{T_Q}. In any case, G^{T_Q} does not have an HP.

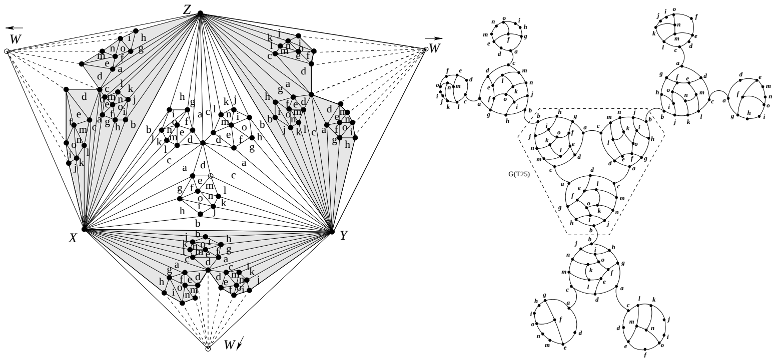


Fig. 6. Case 1: A inner surface viewed from apex w, and the dual graph of the tetrahedralization with pulling apex w.

3 Constructing a convex n-vertex polyhedron P with a constant-length longest path in G^{T_P}

In this section, we shall construct a convex n-vertex polyhedron P such that the longest path in the dual graph of T_p using pulling has constant length in terms of number of vertices n. We can always patch a basic part or a basic building block to a triangular face of convex polyhedron Q to form a new convex polyhedron

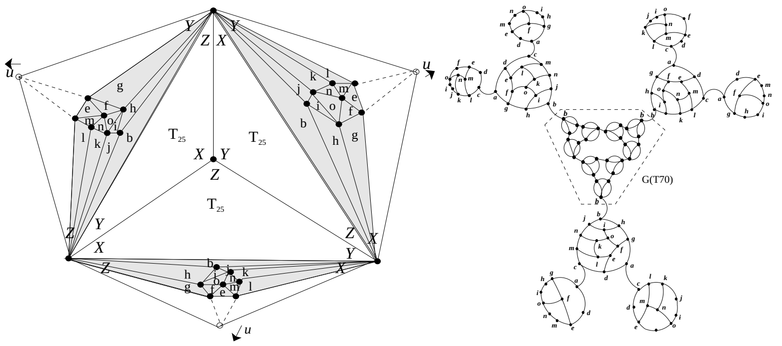


Fig. 7. Case 2(a): A inner surface viewed from apex u, and the dual graph of the tetrahedralization with pulling apex u.

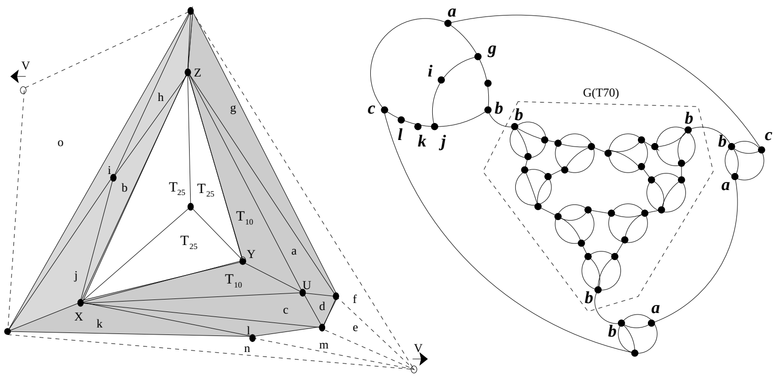


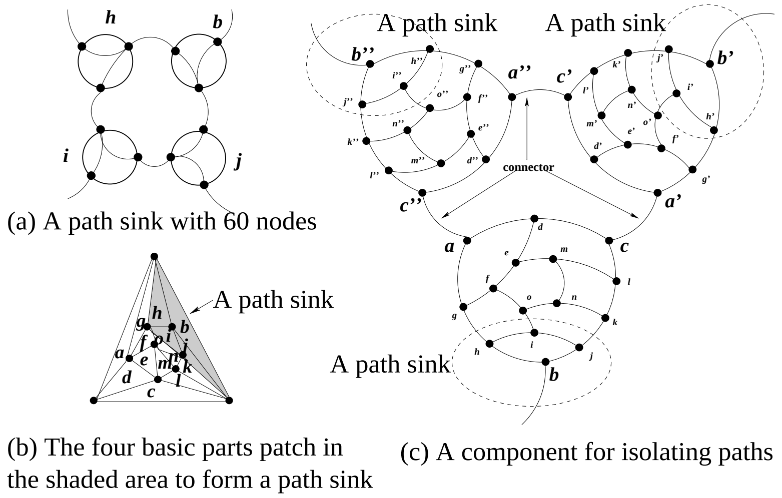
Fig. 8. Case 2(b): inner surface viewed from apex v, and the dual graph of the tetrahedralization with pulling apex v.

with more vertices by Properties 1 and 2. The resulting polyhedron P after n such patch process shall contain $\Theta(n)$ vertices. We shall show that the longest path in G^{T_P} can be as short as $O(1)$ and the maximum number of 'distinct paths' can be as large as $\Theta(n)$. Here, the term 'distinct' means two paths do not share a common node.

To do so, we shall first describe a 3-regular and 3-connected planar graph, called a *path sink*. We then use this path sink to build a component which does not have any HP itself and blocks the paths of any graph that contains this component as a subgraph.

It is easy to check that any path entering the path sink must leave one end there (refer to part (a) of Figure 9). The graph in part (c) of Figure 9 shows that no path can escape the $G^{T_{109}}$ and no path can enter to the center of $G^{T_{109}}$ from outside through nodes b. This is a basic component in the desired dual graph.

To construct the desired convex polyhedron P, we replace the trianglar faces: b, j, i, and h in each T_{10} of T_{25} of Q by basic parts. The four basic parts are so



(a) A path sink with 60 nodes

(b) The four basic parts patch in the shaded area to form a path sink

(c) A component for isolating paths

Fig. 9. The path sink graph contains 60 nodes. The graph $G^{T_{109}}$ consists of a $G^{T_{25}}$ with three path sinks.

placed that their dual graph form a path sink. The new triangulation face on old T_{25} now contains 109 vertices and its dual graph $G^{T_{109}}$ is a basic component. The above patch process can be continued as many as n times due to Property 2 and the resulting polyhedron P is convex by Property 1. Now P has at least $(109-25) \times n$ $(= \Theta(n))$ vertices.

Note that a component $G^{T_{109}}$ requires two paths to be covered. If the path starts at inside $G^{T_{109}}$, then the longest path can cover at most two path sinks and two $G^{T_{10}}$'s, which has 142 nodes. If the path starts at outside a $G^{T_{109}}$, then the longest path can cover at most two path sinks, which has 120 nodes. Thus, the resulting P may have $\Theta(n)$ distinct longest paths and each path has a constant length.

Theorem 2. *There exists a convex polyhedron P with n vertices in which pulling does not yield a Hamiltonian tetrahedralization. The longest path in G^{T_P} covers 142 nodes and the maximum number of distinct longest paths is $\Theta(n)$.*

Corollary 1. *There exists a 3-regular and 3-connected planar graph of 94 nodes which does not have an HP.*

4 conclusion

In this note, we constructed a convex polyhedron such that any tetrahedralization of this polyhedron produced by the pulling method contains no HP in its dual graph. We further showed that the longest path in such a dual graph

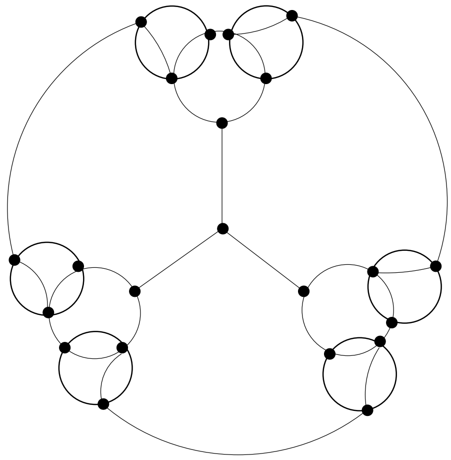


Fig. 10. The 3-regular and 3-connected graph G of 94 nodes contains no HP.

with $\Theta(n)$ nodes may be as short as $O(1)$. We also described a 3-regular and 3-connected planar graph of 94 nodes which does not have an HP.

The obvious open problem is whether or not there is an HP for any tetrahedralization of an arbitrary convex polyhedron by shelling.

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