Information Multicast in (Pseudo-)Planar Networks: Efficient Network Coding over Small Finite Fields

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***Abstract*—Network coding encourages in-network mixing of information flows for enhanced network capacity, particularly for multicast data dissemination. This work aims to explore properties in the underlying network topology for efficient network coding solutions, including efficient code assignment algorithms and efficient encoding/decoding operations that come with small base field sizes. The following cases of (pseudo-)planar types of networks are studied: outer-planar networks where all nodes co-locate on a common face, relay/terminal co-face networks where all relay/terminal nodes co-locate on a common face, general planar networks, and apex networks.**

I. Introduction

Departing from the classic store-and-forward paradigm of data networking, network coding encourages the mixing of information flows within the middle of a network [1] [2], for enhanced network capacity, efficiency, and robustness. Network coding has been extensively studied for one-to-many multicast data dissemination, where it appears particularly beneficial [1] [3]. For a source S multicasting h information flows to a set of receivers T, Li *et al.* [4] proved that linear coding over a sufficiently large finite field is sufficient to achieve the optimal throughput: each link in the network G carries a linear function of the h source flows, and each receiver recovers the h original flows from h linearly independent encoded flows.

Two fundamental problems in multicast network coding are therefore (i) choosing a finite field to perform the encoding/decoding operations in, and (ii) deciding where and how to encode, and hence what information flow to transmit on each link. The latter is known as the *code assignment* problem. Existing literature on network coding often takes an algebraic approach that treats the topology of the network as a blackbox, and designs network coding solutions in a general fashion. For instance, the algebraic framework of Koetter and Médard assumes ubiquitous coding in the network regardless of the topology, and performs code assignment through value assignment that makes the network polynomial non-zero, over the field $GF(2^m)$ with m up to $\lceil \log_2(kh+1) \rceil$ [2], where k denotes the number of receivers. The deterministic code assignment algorithm of Jaggi *et al.* [5] improves the required field size to the same as the number of receivers k, and has a polynomial time complexity. The randomized/non-coherent network coding approach [6] [7] applies randomly selected encoding operations at each node, regardless of its location in the network. It requires a relatively large field for avoiding linearly dependent flows at a receiver.

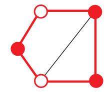
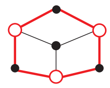
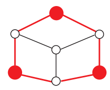
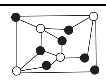
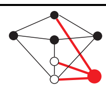
The necessity, benefit and complexity of network coding are indeed sensitive to the specific structure of the network [8] [9] — after all, network coding is coding performed *within a network*. By exploiting the underlying structure of the network topology, one can achieve efficient network coding in many realworld networking scenarios. *Efficient network coding* here is intended to have a two-fold meaning. First, it operates over a very small field, such that the encoding and decoding complexity is minimized [10] [11]. Second, the code assignment algorithm is deterministic, guaranteed to succeed, yet has a linear time complexity, the best asymptotic complexity possible.

This work focuses on planar networks and their variations, a classic subject of study in graph theory and theoretical computer science [12]. Planar networks also have strong connections with real world networking. For instance, Internet backbone networks naturally exhibits a planar embedding on the surface of earth. Furthermore, for non-planar networks such as a dense wireless sensor network, extracting a planar mesh backbone from the network for running network algorithms provides improved efficiency. Unless otherwise stated, we assume the fundamental case of multicast where two source flows are disseminated, which may include an arbitrary number of receivers, requires unbounded field sizes in general networks [2] [5] and leads to the largest known throughput benefit of network coding [13].

Table. I summarizes the main results of this work.

For *outerplanar networks* [12], a special type of planar networks with a face adjacent to all nodes, we prove that network coding and routing (tree packing) are equivalent. We further extract from the proof a linear time algorithm for packing multicast trees, which is NP-hard in general networks [3]. For *relay-coface* networks, planar networks where relay nodes reside on a common face, we prove that that coding over $GF(2)$ is sufficient, and present a linear time algorithm for code assignment over $GF(2)$. For the complementary case of *terminal-coface networks* (multicast source and receivers reside on a common face), we provide a partial proof of a similar result. For general planar networks, we present the first planar networks that require coding over $GF(3)$, and prove that $GF(3)$ is also sufficient. Code assignment algorithms are designed with linear time complexity over $GF(3)$ and quadratic time complexity over $GF(4)$. For *apex networks*,

TABLE I
SUMMARY OF RESULTS

Category	Illustration†	Field Size	Code Assignment Complexity
Outer-planar Networks		tree packing suffices	O($\|V\|$)
Relay-coface Networks		$q = 2$	O($\|V\|$) over $GF(2)$
Terminal-coface Networks		$q = 2$	O($\|V\|$) over $GF(2)$‡
Planar Networks		$q = 3$	O($\|V\|^2$) over $GF(3)$ O($\|V\|$) over $GF(4)$
Apex Networks		$q = 4$	O($\|V\|^2$) over $GF(4)$ O($\|V\|$) over $GF(5)$

†: Solid nodes are multicast terminals, hollow nodes are relays.
‡: Partially proven.

networks that are planar with the removal of a node, we prove that depending on whether the new apex node is a terminal or a relay, the necessary field size can be bounded by 3 and 4, respectively. A code assignment algorithm is designed, with quadratic time using $GF(4)$ or in linear time using $GF(5)$.

In the rest of the paper, we present models and preliminaries in Sec. II. Sections III-VII contain detailed studies of outerplanar, relay-coface, planar, apex and terminal-coface networks, respectively. Sec. VIII concludes the paper.

II. NETWORK MODEL AND PRELIMINARIES

We consider an undirected multicast network $G = (V, E)$ with unit-capacity edges, allowing multiple edges between a pair of nodes. A source S wishes to multicast h information flows to a set of k receivers $T = \{T_1, \ldots, T_k\}$. When $h = 2$, two unit flows x and y are to be disseminated. A link can transmit either x, y or their linear combination. The multicast source and receivers are jointly referred to as the *terminal nodes*. Other nodes in G are *relay nodes*. We assume that the network permits a *static linear algebraic code*, and ignore cases where a convolutional code [14] is required.

For $h = 2$, we assume G is a *minimal network* supporting $h = 2$ [15], *i.e.*, any edge removal makes a multicast rate 2 infeasible. For such edge-minimal networks, we sometimes consider its orientation in which the max-flow from the source to each receiver is 2, and refer to the in-degree and out-degree of nodes in such an orientation. Each relay node is assumed to have degree at least 3; otherwise it can be contracted without affecting the throughput h or the network coding scheme.

A minimal multicast network for $h = 2$ can be decomposed into a set of subtrees, along each of which x, y or a linear function of x and y propagates [16]. A node in the multicast flow is a root of a subtree if it either is the source or has in-degree 2. If the multicast flow f is planar, each subtree forms its own face in a plane embedding of f, as shown in Fig. 1.

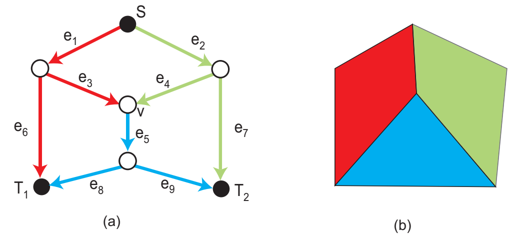


Fig. 1. Illustration of the subtrees and the subtree faces that correspond to a planar multicast flow.

Subtree roots and leaves form the inter-subtree boundaries and are referred to as the *boundary nodes*. A subtree face is an *in-face* for its leaf nodes and an *out-face* for its root. The graph formed by the subtree faces is *a subtree graph*.

III. OUTERPLANAR NETWORKS

A planar graph can be embedded with any of its face being the outer infinite face. For outerplanar graphs, the infinite face is usually chosen to be the common face containing all nodes. Links in an outerplanar networks are categorized into two types: *boundary links* on the infinite face, and *chords* inside.

A. The Equivalence between Network Coding and Routing in Outerplanar Networks

Theorem 1. *In an outerplanar multicast network, with $h = 2$, network coding is equivalent to routing.*

Proof: We present a constructive proof to the theorem, by designing a routing solution (code assignment using the two original flows x and y only) in the following five steps.

1. Pre-processing. An outerplanar network has no K_4 minor [12]. A graph H is a minor of another graph G if H can be obtained from G by a series of link deletion and link contraction operations. By *Dirac's theorem* [17], *every graph with minimum degree 3 contains a K_4 minor*, we can claim that the network must contain a degree-2 node, which must be a terminal, since relay nodes have degree at least 3 (Sec. II). By the *source independence* for network coding in undirected networks [18], we can always switch the multicast source to this degree-2 terminal, without affecting the throughput h or the code assignment.

2. Constructing a Subtree Graph. As described in Sec. II, decompose the multicast network into subtree faces.

3. Forming two Regions. Traverse all boundary nodes in the subtree graph. If a boundary node v has two in-faces only (corresponding to a receiver in the multicast flow without outgoing flows), merge its two adjacent faces F_α and F_β into a new face $F_\alpha \cup F_\beta$ (Fig. 2). Later on we will color the subtree faces. A proper coloring to the merged faces can be converted to a coloring of the original subtree faces: let F_β inherit the color of $F_\alpha \cup F_\beta$, then pick the complementary color for face F_α, who has only one neighbor face.

After the above face merging, we traverse all boundary nodes for a second round. For each boundary node u, label the two of u's adjacent faces that are neighboring the infinite face

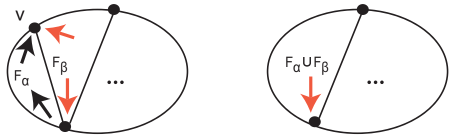


Fig. 2. Merging subtree faces.

as *region 1 faces*, as shown in Fig. 3. All links not included in region 1 (must be chords) are in *region 2*.

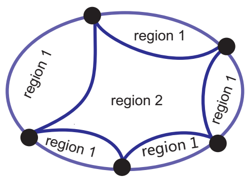


Fig. 3. Two regions to be colored separately.

4. Coloring Region 1. We color the faces in region 1 along the infinite face boundary using two colors x and y. If a boundary node has two flows arriving from two faces in region 1, which intersect only at this node, then expand it into a node pair connected by a new link, (Fig. 4). Such a link expansion is preparing for coloring the subtree faces in region 1, for ensuring flows arriving at the same receiver are independent.

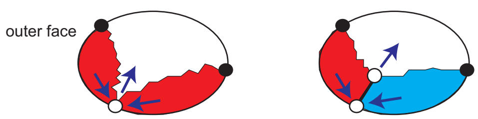


Fig. 4. Expanding a node into two nodes connected by a link, while preserving the planarity of the network.

A potential conflict arises due to the availability of only two colors, if the number of region 1 faces is odd, and each of them share an expanded boundary with each of its two neighbor faces. However, this is impossible for the following reason. For any expanded boundary node v_1, let v_2 and v_3 be the other boundary node for the face on its left and right, respectively. The left and right face are an out face for v_2 and v_3 respectively. At least one of v_2 and v_3 is not the multicast source, and is not expanded.

5. Coloring Region 2. A chord in region 2 cannot enter an expanded boundary node — otherwise that boundary node would have three incoming flows, contradicting $h = 2$. When the chord enters a non-expanded boundary node u, u has at most 1 incoming flow assigned already, and we can pick a color complementary to that flow for the chord link. □

B. Code Assignment Algorithm for Outerplanar Networks

Algorithm 1) is for code assignment in outer-planar networks, extracted from the proof above. It indeed degrades into a tree packing algorithm since network coding is not necessary.

Line 1 can be done by traversing all nodes and links in the network once. Line 2 traverses all faces in region 1. Lines 3-6 traverse edges in region 2. By Euler's formula [12], the number of links and number of faces are both linear in the number of nodes in a planar graph. Therefore, the overall time complexity of Algorithm 1 is $O(|V|)$. We can therefore solve the tree packing problem in linear time, in contrast to the NP-hard complexity of the general tree packing problem [3].

Algorithm 1: Steiner Tree Packing in Outer-planar Networks

Input: an outer-planar multicast network G with $h = 2$
Output: two edge-disjoint Steiner trees.

1 Decompose G into subtree faces. (Sec. III)
// ***Coloring Region*** 1
2 Color the collection of boundary subtree faces neighboring the infinite face greedily with 2 colors x, y.
// ***Coloring Region*** 2
3 **for** *each chord* $\overrightarrow{uv}$ **do**
4 **if** the other incoming edge to v has been colored **then**
5 Color $\overrightarrow{uv}$ with the complement color.
6 **else** Assign x to $\overrightarrow{uv}$.

IV. Relay-coface Networks

We next consider planar networks with relay nodes co-located on the same face. For instance, the wellknown combination network [13] $C_{3,2}$ can be embedded with the three relay nodes on the outer-face. We assume the network G is directed acyclic, for this section only. We prove that $GF(2)$ is sufficient in such a *relay-coface* multicast network.

A. Sufficiency of $GF(2)$ in Relay-coface Networks

We use G_{sub} to denote a *sub-network* of G, containing an induced subset of terminals from G. We first prove the following lemma.

Lemma 1. *Given a planar sub-network G of a combination network $C_{n,2}$, $GF(2)$ is sufficient to achieve multicast rate* 2.

Proof: We construct from G a new graph G_1, and prove G_1 is 3-colorable. Then we map the 3-coloring to a code assignment in G over $GF(2)$. As G is a sub-network of $C_{n,2}$ for some n, a receiver in G is connected to two relay nodes and has out-degree zero. For each receiver in G, contract it with any one of its two neighbors. Let G_1 be resulting graph after such contractions. We show that G_1 is 3-colorable, and any two relay nodes connected to a receiver will receive different colors as they are adjacent in G_1. G_1 doesn't contain a K_4 minor; otherwise, combining the minor in G_1 with the source S, we obtain a K_5 minor in G, contradicting the planarity of G (a planar graph has no K_5 minor [12]). Without a K_4 minor, G_1 must be a series parallel graph and is 3-colorable [19]. Given a proper 3-coloring to the relay nodes, we map the three colors to $\{x, y, x+y\}$. Each receiver receives two different flows from two relays and can recover x and y. □

Theorem 2. *In a planar relay-coface network G, $GF(2)$ is sufficient for multicasting two information flows.*

Proof: For each node v in G, if v is a relay node with exactly one incoming edge, then remove the edge and connect

v directly to the source S if not already so; if v obtains at least one unit information from a node with in-degree 2 or the source, then remove v and the edges incident to v. As all relay nodes are on the same face, connecting some of them to S will not affect planarity. Hence we obtain a bipartite network G' that is also planar, with the relay nodes with in-degree 1 form a group, and the nodes with in-degree 2 form the other group. S is only connected to the relay nodes with in-degree 1. Any two relay nodes with in-degree 1 are not connected in G', otherwise they are connected in G and we can contract the two nodes without affecting the multicast rate 2. Any two terminals are not connected in G' due to the second rule in constructing this bipartite network.

G' is a sub-network of $C_{n,2}$ by construction. Treat each node with in-degree 2 in G' as a receiver. Following the steps in Lemma 1, we obtain a feasible network code over G' with field size $GF(2)$. For edges and nodes existing in G', keep the flows unchanged in G. The key obstacle is to recover the flows which don't exist in G'. From the construction of G', we need to recover the information for the nodes with in-degree 1 and nodes with two incoming flows, one of which comes from a node with in-degree 2.

First, if v is a relay node with in-degree 1 in G, assign the same information it has obtained over G'. Second, if v is the removed nodes with in-degree 2, it must receive one flow from a node u with in-degree 2. Wu *et al.* [20] proved that links entering multicast receivers don't require encoding. Therefore, if v's other flow is from a node with in-degree 1, u can just forward one of its two incoming flows to v, ensuring that v receives two different flows. Otherwise, the two incoming flows with respect to v both come from nodes with in-degree 2, then let v's parents forward two different flows to v. □

From the proof, it is evident that the "relay-coface" condition can be relaxed, and the sufficiency of $GF(2)$ holds as long as that all relay nodes with in-degree 1 are on the same face. This extends the applicability of the theorem to, for example, the classic butterfly network, which has three relay nodes with in-degree 1, all of which reside on the infinite face.

B. Code Assignment Algorithm for Relay-coface Networks

An efficient code assignment algorithm for relay-coface networks can be extracted from the proof to Theorem 2, as shown in Algorithm 2.

The algorithm consists of three main steps. The first step (lines 1-5) of constructing G' can be done by traversing all nodes in G. The second step (lines 6-9) constructs a network code in this bipartite network G'. Here the complexity mainly depends on the coloring algorithm in G_1, which takes linear time [19]. In the last step (lines 10-18), we need to recover the flows for the removed nodes in G. This requires visiting all the nodes and the incoming edges with respect to them. As a node has at most two incoming edges, this step can be finished in linear time in the number of nodes. To conclude, Algorithm 2 has a linear time complexity.

Algorithm 2: Code Assignment Algorithm for Relay-coface Networks

Input: a relay-coface multicast network G, with $h = 2$.
Output: a valid code assignment scheme over $GF(2)$.

// ***Constructing a Bipartite Network*** G'
1 **for** *each v with in-degree 1* **do**
2   Remove its incoming edge and connect it to the source S.
3 **for** *each v with in-degree 2* **do**
4   **if** $\exists$ *a parent u, s.t. in-degree(u) =2* **then**
5   Remove v and all the edges incident to v.

// ***Design a Network Code in*** G'
6 **for** *each node with in-degree 2 in* G' **do**
7   Contract it with one of its parents.
8 Remove the source S and let G_1 denote the graph obtained here.
9 Color the graph G_1 with 3 colors [19].

// ***Design a Network Code in*** G
10 **for** *each node v with in-degree* 1 **do**
11   Assign it the same flow that it has obtained in G'.
12 **for** *each v with in-degree 2 in G and* $v \notin G'$ **do**
13   Assume its two parents are u_1, u_2.
14   **if** in-degree(u_1)=2 and in-degree(u_2)=2 **then**
15   u_1, u_2 forward two different flows they have received to v.
16   **else if** in-degree(u_1)=2 **then**
17   u_1 forwards a flow to v different from the other incoming flow of v.
18   **else** u_2 forwards a flow to v different from the other incoming flow of v.

V. General Planar Networks

A. The Necessity of $GF(3)$ in General Planar Networks

Interestingly, we are not aware of an example multicast network in the literature that both has a planar topology, and requires coding over $GF(3)$. We design new multicast networks to show that coding over $GF(3)$ is indeed necessary in general planar networks.

Theorem 3. *There exist planar networks that require* $GF(3)$ *for achieving the optimal multicast throughput.*

Proof: Fig. 5 depicts a multicast network, with one source S and five receivers $\{T_1, T_2, T_3, T_4, T_5\}$. Each link has a unit capacity. To achieve throughput 2, there is only one possible network orientation.At least 4 linearly independent flows are required to satisfy the demands of receivers $T_i, i = 1, 2, 3, 4, 5$, hence a minimum field size of 3 is necessary in this network.

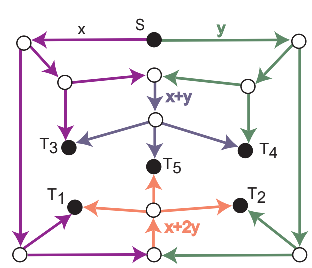


Fig. 5. A planar multicast network in which multicast rate 2 can be achievable by coding over $GF(3)$ but not over $GF(2)$.

Theorem 4. *There exist bipartite planar networks that require $GF(3)$ for achieving the optimal multicast throughput.*

Proof: Fig. 6 shows a bipartite multicast network with one source, four relay nodes and six receivers. A throughput of 2 is feasible only if the base field size is at least 3. □

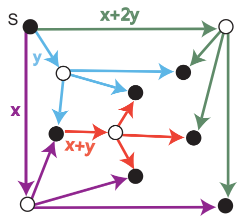


Fig. 6. A planar bipartite multicast network requiring $GF(3)$.

B. The Sufficiency of $GF(3)$ in General Planar Networks

Theorem 5. *Coding over $GF(3)$ is sufficient for multicasting two information flows in a planar network.*
Proof: We first compute a multicast flow achieving throughput two [3] and construct a subtree graph as described in Sec. II. If the two in-faces of a node u intersect only at this node, we expand u into a node pair, connected by a new link (Fig. 4). By the Four Color Theorem [12], every planar graph can be colored using four colors, such that no two adjacent faces share a common color. Two faces are adjacent if they share a common link, and are not adjacent if they share only a common node. We color the subtree faces processed in step 2, using the following four colors: x, y, $x+y$ and $x+2y$.

To verify that the code assignment is valid, it is sufficient to show that the root u of each subtree face has two distinct incoming flows. This is true since u has two in-faces due to the property of the subtree decomposition. Furthermore, the expansion operation in Step 2 guarantees that these two in-flows always share a common boundary, and hence will be assigned different colors/flows. Once receiving two distinct flows, the root u is able to linearly combine them for generating the flow assigned to its out-face(s), if any. To conclude, we obtained a valid code assignment over $GF(3)$, for achieving multicast throughput $h=2$ in the planar network. □

C. Code Assignment Algorithm in General Planar Networks

Algorithm 3: Code Assignment Algorithm for General Planar Networks

Input: a planar multicast network G, with $h=2$.
Output: a valid code assignment over $GF(3)$ ($GF(4)$).

Decompose G into subtree faces. (Sec. III)
for *each node v with in-degree* 2 **do**
    if *two subtrees faces with v as a leaf only intersect at v* **then**
        Expand v to an edge shared by these two faces (Fig. 4).
Color the subtree faces with four colors [21] or five colors [22].

Algorithm 3 is extracted from the proof to Theorem 5. Its overall computational complexity is dominated by face coloring the subtree graph. With four colors (coding over $GF(3)$), it takes $O(|V|^2)$ time to four-color a planar graph [21]. However, if we encode over a larger field $GF(4)$, then five-coloring can be accomplished in $O(|V|)$ time [22]. $GF(4)$ is perhaps preferred in realworld implementations, for the following three reasons: (1) efficient linear time code assignment; (2) exact 2-bit representation of a symbol; and (3) efficient implementation of addition, which is equivalent to bit-wise XOR.

VI. Apex Networks

A pseudo-planar network that can be made planar by removing a single vertex (the *apex node*) is an *apex network* [23]. For example, Fig. 7(a) shows a planar multicast network that requires coding over $GF(2)$. By adding an extra receiver node v, we obtain an apex network in Fig. 7(b) that requires coding over $GF(3)$.

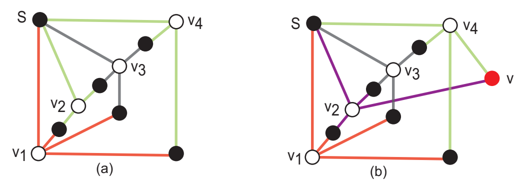


Fig. 7. A planar multicast network and an apex multicast network.

Theorem 6. *In an apex network G, with $h=2$, coding over $GF(4)$ suffices.*
Proof: We prove the sufficiency of $GF(4)$ by examining each of the three cases of the in-degree of the apex node v.

1. in-degree(v) = 0. Then v must be the multicast source. Let N_v be its neighbors. Pretend that the nodes $\in N_v$ in $G-v$ have the same in-degree as in G. Starting from nodes in N_v, construct the subtree graph over $G-v$ based on the in-degrees. Perform the expanding operation when needed and then apply the four-coloring algorithm [21] to color the subtree faces. As v is the source, it can transmit any flow from {x, y, x+y, x+2y} to N_v. For each node u in N_v over G, assign the flow that u has obtained through $\overrightarrow{vu}$ over $G-v$ to the edge $\overrightarrow{vu}$. For other flows over G, keep them the same as assigned in $G-v$.

2. in-degree(v) = 1. In this case, there is a subtree $\mathcal{T}$ over G containing all the edges incident to v. Assign a new color x+3y to $\mathcal{T}$. For the remaining subtrees, they form a planar graph which is a subgraph of $G-v$, and can be four-colored.

3. in-degree(v) = 2. If v is a relay node, change it to a receiver role. Li and Li [18] proved the source independence property that states the multicast throughput and code assignment can be independent of the selection of the source within the multicast group. We can exchange the source and receiver roles between the original sender S and v. Then refer to the first case for the proof of sufficiency of $GF(3)$. □

From the proof above, we can indeed see that the increment of the field size from $GF(3)$ to $GF(4)$ is only required when the apex node is a relay, and is not necessary when the apex node is a terminal. The complexity of code assignment procedure described in the proof is dominated by the face coloring of a graph obtained from G. If we perform four-coloring, the overall field size requirement is $GF(4)$ and the time complexity is $O(|V|^2)$. If we perform five-coloring, the field size requirement is $GF(5)$ and the time complexity is $O(|V|)$. If we further know that the apex node is a terminal, the required field size decreases by 1 in both cases.

VII. Terminal-coface Networks

In this section, we provide a partial proof to the conjecture that $GF(2)$ suffices for terminal-coface networks. For the subtree graph of a multicast network G, contract each subtree to a node and connect the two nodes if the two corresponding subtrees share a common leaf in G. We refer to the resulting graph as the *subtree-node graph* of G. A recent work of Yin *et al.* [9] show that if a multicast network G requires $GF(3)$, then the subtree-node graph of G must contain a K_4 minor. Below we first prove that the subtree-node graph of a terminal-coface network can not be K_4.

Lemma 2. *If a planar multicast network G's subtree-node graph H is K_4, then G is not a terminal-coface network.*

Proof: Embed G with the terminals on the outer face. We first prove that when constructing the subtree-node graph H, the nodes corresponding to the subtrees with a receiver as a leaf form the outer face of H. Then we show H can't be K_4.

In the process of constructing H, it's sufficient to perform only link contraction. If two subtrees share a common receiver as a leaf, contract the receiver with one of the two subtree-nodes to produce the edge connecting them in H. As the receiver is on the outer infinite face of G, after the contraction, the edge between the two subtree-nodes is still incident to the outer face. For the subtrees containing terminals, the nodes corresponding to them are also incident to the outer face.

Next, if H is K_4, let the boundary of its outer-face be defined by three nodes v_1, v_2, v_3. From the above, we know that the remaining subtree node v_4 doesn't contain any receiver as a leaf, otherwise it should be on the same face with v_1, v_2, v_3. For each edge $v_iv_4, i = 1, 2, 3$ in H, v_i and v_4 share a non-receiver node with in-degree 2 and must produce a different subtree node. There are at least two subtree nodes on the outer face, say, v_1 and v_2, with the source as the root. Then no subtree can have its root be a common leaf of v_3 and v_4, contradiction. □

Theorem 7. *Let G be a terminal co-face planar network with $h = 2$ and subtree-node graph H. If H has maximum node degree 3, then $GF(2)$ is sufficient for G.*

Proof: From Lemma 2, the subtree-node graph H cannot be K_4. Moreover, *Brooks's theorem* [24] states that every connected graph with the maximum vertex degree at most three has a 3-coloring, otherwise it is isomorphic to K_4. Therefore, H can be properly colored in 3 colors, and $GF(2)$ is sufficient for coding in G. □

VIII. Conclusion

We studied multicast network coding in a series of special planar, planar and pseudo-planar networks in this work, including outerplanar networks, relay/terminal co-face networks, planar networks, and apex networks. We prove the no coding at all, or coding over very small finite fields, suffices in these cases, and further extract efficient code assignment algorithms from the constructive proofs. An interesting future work is to generalize the results from two source information flows to an arbitrary number of source flows.

References

[1] R. Ahlswede, N. Cai, S. Li, and R. Yeung, "Network Information Flow," *IEEE Transactions on Information Theory*, vol. 46, no. 4, pp. 1204–1216, 2000.

[2] R. Koetter and M. Médard, "Beyond Routing: An Algebraic Approach to Network Coding," in *Proc. of IEEE INFOCOM*, 2002.

[3] Z. Li, B. Li, D. Jiang, and L. Lau, "On Achieving Optimal Throughput with Network Coding," in *Proc. of IEEE INFOCOM*, 2005.

[4] S. Li, R. Yeung, and N. Cai, "Linear Network Coding," *IEEE Transactions on Information Theory*, vol. 49, no. 2, pp. 371–381, 2003.

[5] S. Jaggi, P. Sanders, P. Chou, M. Effros, S. Egner, K. Jain, and L. Tolhuizen, "Polynomial Time Algorithms for Multicast Network Code Construction," *IEEE Transactions on Information Theory*, vol. 51, no. 6, pp. 1973–1982, 2005.

[6] M. Jafari, S. Mohajer, C. Fragouli, and S. Diggavi, "On the Capacity of Non-Coherent Network Coding," in *Proc. of IEEE International Symposium on Information Theory*, 2009.

[7] T. Ho, M. Médard, J. Shi, M. Effros, and D. Karger, "On Randomized Network Coding," in *Proc. of the 41st Allerton Conference*, 2003.

[8] J. Ebrahimi and C. Fragouli, "Properties of Network Polynomials," in *Proc. of IEEE International Symposium on Information Theory*, 2012.

[9] X. Yin, Y. Yang, X. Wang, X. Yang, and Z. Li, "A Graph Minor Perspective to Network Coding: Connecting Algebraic Coding with Network Topologies," in *Proc. of IEEE INFOCOM*, 2013.

[10] M. Lanberg and A. Sprintson, "Recent Results on the Algorithmic Complexity of Network Coding," in *Proc. of the 5th Workshop on Network Coding, Theory, and Applications*, 2009.

[11] R. Dougherty, C. Freiling, and K. Zeger, "Linearity and Solvability in Multicast Networks," *IEEE Transactions on Information Theory*, vol. 50, no. 10, pp. 2243–2256, 2004.

[12] R. Diestel, *Graph Theory*. Springer-Verlag, 1997.

[13] S. Maheshwar, Z. Li, and B. Li, "Bounding The Coding Advantage of Combination Network Coding in Undirected Networks," *IEEE Transactions on Information Theory*, vol. 58, no. 2, pp. 570–584, 2012.

[14] S. Jaggi, M. Effros, T. Ho, and M. Médard, "On Linear Network Coding," in *Proc. of the 42nd Allerton Conference*, 2004.

[15] S. E. Rouayheb, C. Georghiades, and A. Sprintson, "Network Coding in Minimal Multicast Networks," in *Proc. of IEEE Information Theory Workshop*, 2006.

[16] C. Fragouli, E. Soljanin, and A. Shokrollahi, "Network Coding as a Coloring Problem," in *Proc. of the 38th Annual Conference on Information Sciences and Systems*, 2004.

[17] G. Dirac, "A Property of 4-Chromatic Graphs and Some Remarks on Critical Graphs," *London Mathematical Society*, vol. 1, no. 1, p. 85, 1952.

[18] Z. Li and B. Li, "Network Coding in Undirected Networks," in *Proc. of the 38th Annual Conference on Information Sciences and Systems*, 2004.

[19] P. Seymour, "Colouring Series-Parallel Graphs," *Combinatorica*, vol. 10, no. 4, pp. 379–392, 1990.

[20] Y. Wu, K. Jain, and S. Kung, "A Unification of Network Coding and Tree-Packing (Routing) Theorems," *IEEE Transactions on Information Theory*, vol. 52, no. 6, pp. 2398–2409, 2006.

[21] N. Robertson, D. Sanders, P. Seymour, and R. Thomas, "Efficiently Four-Coloring Planar Graphs," in *Proc. of the 28th ACM Symposium on Theory of Computing*, 1996.

[22] G. Frederickson, "On Linear-Time Algorithms for Five-Coloring Planar Graphs," *Information Processing Letters*, vol. 19, no. 5, pp. 219–224, 1984.

[23] D. Eppstein, "Diameter and Treewidth in Minor-Closed Graph Families," *Algorithmica*, vol. 27, no. 3, pp. 275–291, 2000.

[24] R. Brooks, "On Colouring the Nodes of a Network," in *Proc. of Mathematical Proceedings of the Cambridge Philosophical Society*, 1941.