# Online Inserting Points Uniformly on the Sphere

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**Abstract.** In many scientific and engineering applications, there are occasions where points need to be inserted uniformly onto a sphere. Previous works on uniform point insertion mainly focus on the offline version, i.e., to compute $N$ positions on the sphere for a given interger $N$ with the objective to distribute these points as uniformly as possible. An example application is the Thomson problem where the task is to find the minimum electrostatic potential energy configuration of $N$ electrons constrained on the surface of a sphere. In this paper, we study the online version of uniformly inserting points on the sphere. The number of inserted points is not known in advance, which means the points are inserted one at a time and the insertion algorithm does not know when to stop. As before, the objective is achieve a distribution of the points that is as uniform as possible at each step. The uniformity is measured by the gap ratio, the ratio between the maximal gap and the minimal gap of any pair of inserted points. We give a two-phase algorithm by using the structure of the regular dodecahedron, of which the gap ratio is upper bounded by 5.99. This is the first result for online uniform point insertion on the sphere.

## 1 Introduction

In this paper, we consider the problem of inserting points onto the sphere such that the inserted points are as uniformly spaced as possible. There are many applications, e.g., the *Thomson problem* [13] which was introduced by the physicist Sir Joseph John Thomson in 1904; the objective is to determine the configuration of $N$ electrons on the surface of a unit sphere that minimizes the electrostatic potential energy, which translates directly into the problem of placing $N$ points on the surface of the sphere as uniformly as possible. The minimum energy configuration of the Thomson problem and other configurations with uniform point distribution on the unit sphere play important roles in many scientific and engineering applications [3, 6, 8, 9], e.g., 3D projection reconstruction of Computed Tomography (CT) or Magnetic Resonance Images (MRI).

From the perspective of computer science, the traditional Thomson problem is offline, i.e., the number of points is known in advance and the objective is to place these points on the sphere as uniformly as possible. Inserting points in an online fashion is also an interesting problem, which might have its application in real life. An example is to assign an unknown number of volunteers as they show up to an area where a rescue mission is in progress. In the online version, the points are inserted over a time span, and the strategy has no idea about the number of points to be inserted. The position of a point cannot be changed after it has been inserted.

Any solution to the online problem is to be measured by the uniformity of the distribution of the points. There are several ways to define the uniformity of a set of points. Some studies define the uniformity according to the minimal pairwise distance [7, 11]. In discrepancy theory [5, 10], uniformity is defined as the ratio between the maximal and minimal number of points in a fixed shape within the area. In this paper, uniformity is defined to be the gap ratio, which is the ratio between the maximal gap and the minimal gap between any pair of points.

Formally, let $S$ be the surface of a 3-dimensional unit sphere. The task is to insert a sequence of points onto $S$. Let $p_i$ be the $i$-th point to be inserted and $S_i = \{p_1, \ldots, p_i\}$ be the configuration in $S$ after inserting the $i$-th point. In configuration $S_i$, let the maximal gap be $G_i = max_{p \in S} min_{q \in S_i} 2 \cdot \overset{\frown}{d}(p, q)$, the minimal gap be $g_i = min_{p,q \in S_i, p \neq q} \overset{\frown}{d}(p, q)$, where $\overset{\frown}{d}(p, q)$ is the spherical distance between points $p$ and $q$, i.e., the shortest distance along the surface of the sphere from $p$ to $q$. In other words, the maximal gap is the spherical diameter of the largest empty circle while the minimal gap is the minimal spherical distance between two inserted points. We call $r_i = G_i/g_i$ the $i$-th gap ratio. The objective is to insert points onto $S$ as uniformly as possible so that the maximal gap ratio ($\min \max_i r_i$) during the insertion of the whole set of points is minimized.

The problem of uniformly inserting points in a given area has been studied before. Teramoto et al [12] and Asano et al [2] showed that the Voronoi insertion is a good strategy on the plane; moreover, the gap ratio of the the Voronoi insertion is proved to be at most 2. They also studied insertion onto a one-dimensional line; if the algorithm knows the number $n$ of the points to be inserted, an insertion strategy with maximal gap ratio $2^{\lfloor n/2 \rfloor / (\lfloor n/2 \rfloor + 1)}$ can be derived. If the points must be inserted at the fixed grid points, Asano [1] gave an insertion strategy with uniformity 2 for the one dimensional case. For insertion on two-dimensional grid, Zhang et al. [14] proved the lower bound to be at least 2.5 and gave an algorithm with the maximal gap ratio 2.828. Recently, Bishnu et al. [4] considered some variants and measurements of the insertion on the Euclidean space.

In the remainder of this paper, we present a strategy for online inserting points uniformly onto the surface of a sphere with a maximal gap ratio of no more than 5.99. This is the first result for the problem of online insertion of points on sphere.

## 2 Point Insertion Strategy

A simple intuitive idea is to greedily insert the incoming point at the "center" of the largest empty spherical surface area. The early steps of such a greedy approach are simple; however, when many points have been inserted, the shapes of different local structures may vary significantly and the configuration may become very complicated. As a result, the computational cost of finding the largest empty spherical surface area and then computing its center may become prohibitive.

Observe that once some points have been inserted, the sphere is partitioned into local structures and the next point insertion within the area of some local structure will only affect the local configuration, i.e., the spherical distances (including the max gap and min gap) outside this area do not change. Based on this observation, a two-phase strategy can be devised. In the first phase, we use a polyhedron to approximate the sphere and points are inserted at the vertices of the polyhedron. After all vertices of the polyhedron are occupied, the second phase starts. In the second phase, we recursively compute the point positions in all faces of the polyhedron, and the inserted points on the sphere are the projections of these positions onto the sphere.

As mentioned before, the computation cost of the simple greedy approach is large due to the complicated local structures on the sphere when the a large number of points have been inserted. In this paper, a regular dodecahedron is used to simulate the shape of the sphere. A regular dodecahedron has twelve identical regular pentagonal faces and twenty vertices. In the first phase, handling the insertion of twenty points on the sphere is quite straightforward and the gap ratio is not large. The main advantage of the regular dodecahedron lies in the processing of the second phase. Since all faces of the decahedron are identical, we only need to consider how to insert points onto the sphere with respect to a regular pentagon. When the number of points inserted increases, the refinement of the regular pentagon contains local structures which can be categorized as three types (see Section 2.2). Then according to these three types of local shapes, we can impose a recursive procedure to compute the next point insertion positions.

Since changing the radius of the sphere does not affect the gap ratio, for the convenience of computation, we assume that the radius of the sphere is $\sqrt{3}$. Thus, the length of each edge of the corresponding regular dodecahedron is $\frac{4}{\sqrt{5}+1}$. In the following, we give the details of the two phases of our algorithm.

### 2.1 The First Phase

In our strategy, the sphere can be divided into 12 sections by projecting the 20 vertices and all edges of the regular dodecahedron onto the sphere, as shown in Figure 1. On the dodecahedron, eight orange vertices with coordinates (±1, ±1, ±1) form a cube (dotted lines). Let $O$ be the center of the cube and let $\phi = (1+\sqrt{5})/2 \approx 1.618$ be the golden ratio. Four green vertices lying at (0, $\pm 1/\phi$, $\pm\phi$) form a rectangle on the $yz$-plane. Four blue vertices lying at ( $\pm 1/\phi$, $\pm\phi$, 0) form a rectangle on the $xy$-plane. Four pink vertices lying at ($\pm\phi$, 0, $\pm 1/\phi$) form a rectangle on the $xz$-plane.

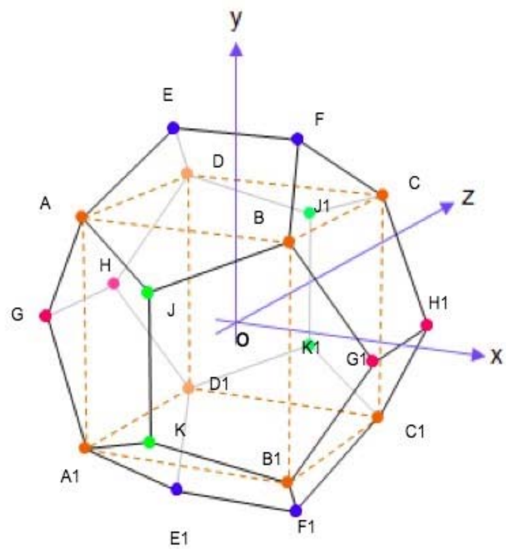



**Fig. 1.** vertex distribution of the regular dodecahedron

The insertion strategy of the twenty points is as follows.

1. First insert eight points at orange vertices with coordinates (±1, ±1, ±1), i.e., the vertices of the cube $(A, B, C, D, A_1, B_1, C_1, D_1)$. The order of the inserted points is $A$, $C_1$, followed by an arbitrary order of the remaining 6 points.
2. Then insert the remaining twelve points in any arbitrary order.

**Lemma 1.** *During the insertion at the first eight vertex points of the dodecahedron, the gap ratio is no more than 2.55.*

*Proof.* After the insertion of two points at $A$ and $C_1$, the maximal gap and the minimal gap are both the spherical distance between these two points, i.e.,

$$G_2 = g_2 = \widehat{d}\ (A, C_1).$$

In this case, the gap ratio $r_2 = 1$.

Note that the radius of the sphere is $\sqrt{3}$. Arbitrarily choose any one of the remaining six points, w.l.o.g., say $B$. After the insertion at $B$, the maximal gap is still $G_3 = \widehat{d}\ (A, C_1) = \sqrt{3}\pi$ while the minimal gap decreases to $g_3 = \widehat{d}\ (A, B)$, which is the value of $\sqrt{3}\angle AOB$. Since $|OA| = |OB| = \sqrt{3}$ and $|AB| = 2$, $\angle AOB = 2 \cdot \arcsin(\frac{\sqrt{3}}{3}) = 1.231$ and $g_3 = 2.132$. Thus, the gap ratio $r_3 = G_3/g_3 = 2.55$.

After inserting any other points in this sub-phase, the value of the minimal gap does not change while the value of the maximal gap may decrease. After all the eight points have been inserted, the maximal gap

$$G_8 = \sqrt{3} \cdot \angle AOB_1 = 3.309$$

while the minimal gap $g_8 = g_3$. Thus at this stage, the gap ratio is $G_8/g_8 = 1.55$.

Hence, the maximal gap ratio for inserting the first eight points is 2.55. $\square$

Now we analyze the gap ratio for inserting the remaining twelve points at the vertices of the dodecahedron.

**Lemma 2.** *During the process of inserting the remaining twelve vertex points of the dodecahedron, the maximal gap ratio for the sphere is at most 2.615.*

*Proof.* W.l.o.g., assume that the first point inserted in this sub-phase is $E$, and thus the minimal gap $g_9 = \widehat{d}\ (A, E)$. At this stage, since all eight points on the cube have been inserted, the maximal gap $G_9 = \widehat{d}\ (A, B_1) = 3.309$.

$$g_9 = \sqrt{3} \cdot 2 \cdot \arcsin \frac{|AE|/2}{R} = \sqrt{3} \cdot 2 \cdot \arcsin \frac{2}{(\sqrt{5}+1)\sqrt{3}} = 1.264.$$

Thus, the gap ratio

$$r_9 = \frac{G_9}{g_9} = 2.618.$$

For the remaining eleven points in this sub-phase, the minimal gap will not decrease while the maximal gap will not increase. Hence, the maximal gap in this sub-phase is at most 2.615. $\square$

**Lemma 3.** *The maximal gap ratio in the first phase is 2.618.*

### 2.2 The Second Phase

After all the vertices on the dodecahedron have been inserted, the second phase begins. As mentioned before, the regular dodecahedron has some good property and consequently, further point insertions can be done recursively on the sphere with respect to the corresponding structures of the faces of the dodecahedron after some points have been inserted onto them. Moreover, in our strategy, we first compute positions on the faces of the regular dodecahedron, and the true insertions will be done at the projected positions of these computed points on the sphere.

By such an implementation, point insertion on the sphere is reduced to the point insertion on the plane faces of the dodecahedron, which is much easier to handle. However, the gap ratio on the plane (pentagon) is smaller than that on the sphere. In the remaining part of this subsection, we first show that the difference of the gap ratios on sphere and on plane is quite small, and then we give the strategy of how to insert points on a pentagon.

**The difference of the gap ratio** W.l.o.g., we consider point insertion on the pentagon $AJBFE$. First, we consider the situation where two inserted points are both on the sphere and on the pentagon. Since only five vertices of the pentagon satisfy such condition, there are two cases to be examined. Let the edge of the pentagon be $\ell$, i.e., $|AJ| = \ell$. Let $O'$ be the center of the pentagon $AJBFE$. Since $|AJ|^2 = |AO'|^2 + |JO'|^2 - 2|AO'| \cdot |JO'| \cos(2\pi/5)$, we have $|AO'| = |JO'| = 0.85\ell$, and $|OO'| = \sqrt{R^2 - |AO'|^2} = 1.11\ell$.

- First, we consider the subsituation that the two inserted points are not adjacent vertices of the pentagon. W.l.o.g., let $A$ and $B$ denote these two inserted points. Since the angle $\angle AJB = 3\pi/5$, we can see that $|AB| = 1.618\ell$. By the property of the regular dodecahedron, the radius of the sphere $R = 1.4\ell$. We then have

$$\overset{\frown}{d}\,(A, B) = R \cdot \angle AOB = R \cdot 2 \cdot \arcsin \frac{|AB|}{2R} = 1.231R.$$

Thus,

$$\frac{\overset{\frown}{d}\,(A, B)}{|AB|} = 1.066.$$

- Then we consider the subsituation that the two inserted points are adjacent vertices of the pentagon. W.l.o.g., let $A$ and $J$ denote these two inserted points. By a similar analysis, we have

$$\overset{\frown}{d}\,(A, J) = R \cdot \angle AOJ = R \cdot 2 \cdot \arcsin \frac{|AJ|}{2R} = 0.73R.$$

Thus,

$$\frac{\overset{\frown}{d}\,(A, J)}{|AJ|} = 1.023.$$

By the above analysis, we can see that for two positions that are on the sphere, the ratio between the spherical distance and the direct distance is monotonically increasing with respect to the corresponding subtended angle.

Next we consider the situation that both of the two inserted points lie inside the pentagon.

As shown in Figure 2, $C$ and $D$ are two points lying inside the pentagon, and $C'$ and $D'$ are their projections on the sphere respectively. Let $|C'D'|$ be the direct distance between $C'$ and $D'$; thus,

$$\frac{|C'D'|}{|CD|} \leq \frac{R}{|OO'|} = 1.26.$$

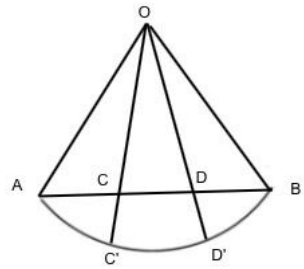



**Fig. 2.** Points on the plane and the corresponding projections on the sphere.

According to the previous description, $\widehat{d}\ (C', D')/|C'D'| \leq \widehat{d}\ (A, B)/|AB|$ since with respect to the sphere, the subtended angle is maximized for $A$ and $B$.

Thus,

$$\frac{\widehat{d}\ (C', D')}{|CD|} = \frac{\widehat{d}\ (C', D')}{|C'D'|} \cdot \frac{|C'D'|}{|CD|} \leq 1.34.$$

For any two spherical distances $\widehat{d}\ (A, B)$ and $\widehat{d}\ (C, D)$, its ratio is upper bounded by

$$\frac{\widehat{d}\ (A, B)}{\widehat{d}\ (C, D)} \leq 1.34 \cdot \frac{|AB|}{|CD|}.$$

Therefore, the comparison between two spherical distances can be reduced to the comparison between two direct distances and the ratio would not change much.

**Insertion on the pentagon** In this part, we describe how to compute the point insertion positions on the pentagon.

The pentagon can be recursively partitioned into smaller polygons of one of three shapes, as shown in Figure 3. For the pentagon $AJBFE$, by connecting non-adjacent vertices, we can see that the pentagon is partitioned into eleven parts, one smaller pentagon $ajbfe$, five isosceles triangles with vertex angle $\frac{\pi}{5}$ ($Aae$, $Jja$, $Bbj$, $Ffb$ and $Eef$), and five isosceles triangles with vertex angle $\frac{3\pi}{5}$ ($AJa$, $JBj$, $BFb$, $FEf$ and $EAe$).

The isosceles triangles can be further partitioned into smaller isosceles triangles of the above two shapes, as shown in Figure 4. For example, in isosceles triangle $AJj$, by adding the point $p_1$, two isosceles triangles, say $Jjp_1$ and $AJp_1$ emerge, which are still of the above two shapes. For isosceles triangle $AeE$, by adding a point $h_1$, $AeE$ is partitioned into two isosceles triangles $Aeh_1$ and $Ah_1E$, both of which are again of the above two shapes.

In the above description, we can see that the pentagon can be recursively partitioned into three types of polygons. Such property can be used in the in-

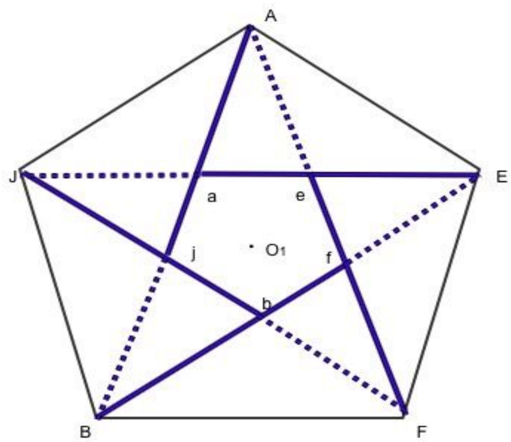
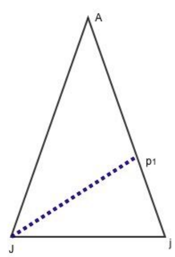
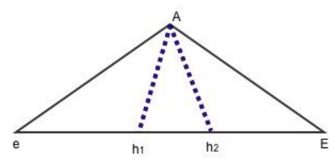



**Fig. 3.** The insertion in the pentagon.




(a) isosceles acute triangle




(b) isosceles obtuse triangle

**Fig. 4.** The insertion in isosceles triangle.

sertion strategy in order to reduce the complexity of the computation.

**Insertion Strategy** In the insertion strategy, three queues $Q_1$, $Q_2$ and $Q_3$ are used to store these three types of shapes; for each type, the sizes of the objects in the corresponding queue are non-increasing, where the size of a polygonal shape is defined to be the length of its longest edge.

When a new point comes in, the following procedure applies.

– Compare the insertions on the heads of these three queues and select the one with the largest minimal gap if the point is inserted at an appropriate position. As shown in Figure 3, the inserted points in pentagon $AJBFE$ can be $a$, $j$, $b$, $f$ or $e$, respectively; as shown in Figure 4(a), the inserted point in the isosceles acute triangle $AJj$ is $p_1$; as shown in Figure 4(b), the inserted point in the isosceles obtuse triangle $AeE$ is $h_1$ or $h2$. Assume that the selected polygon is $P$.

- Determine the point insertion position $x$ on the corresponding polygon $P$.
- Note that $P$ is partitioned into some smaller polygons. Remove $P$ from the head of the queue and then add these smaller polygons at the tail of the corresponding queues.
- Find the point insertion position $X$ which is the projection of $x$ onto the sphere.

Note that when we are processing a triangle, the inserted point is on an edge of the triangle, which is also on the edge of another triangle with the same size and of the same type or the other type. In this case, both of these two triangles will be partitioned and removed from the queue, and the newly created smaller triangles will be added at the tails of the corresponding queues.

**Lemma 4.** *After a point is inserted in the above operation, in each of the three queues, the sizes of the objects are still in non-increasing order.*

*Proof.* This lemma can be proved as follows.

- First, we consider the pentagon. Initially, there are twelve pentagons with the same size. When each of them is processed, a smaller one will appear and the larger one will be removed from the queue. Since the queue is ordered in the initial stage, the order property will hold at any time.
- Then we consider the isosceles triangle. Initially, the queue is empty and thus it is ordered. After the partition of a pentagon or an isosceles triangle, if the order does not hold, i.e., the size of the newly created polygon is larger than the size of the tail polygon in the same queue. This means that the selection criteria is violated. Contradiction!

Hence, this lemma follows. □

**Gap ratio analysis** In this part, we analyze the gap ratio of the above strategy. Since there are three different shapes, we will study all these three cases one by one.

- First, we consider point insertion in a pentagon; the inserted points are shown in Figure 3. Let $O_1$ be the center of the pentagon $AJBFE$, as shown in Figure 3. In this case, the maximal gap is twice of $|O_1A|$, which is the radius of the circumcircle of the pentagon. Thus, at this stage, the maximal gap
$$G = 2 \cdot |O_1A| = 2 \cdot 0.85\ell = 1.7\ell$$
where $\ell$ is the length of the pentagon.
During the point insertion operation, the minimal gap is lower bounded by the length of the smaller pentagon. Thus, the minimal gap $g$ is at least
$$|BE| - |Ba| - |eE| = |BE| - 2 \cdot |Ba|.$$
Note that the length of an edge $x$ of a triangle $XYZ$ can be computed by $x = \sqrt{y^2 + z^2 - 2yz \cos \angle X}$. After computation, we have $|BE| = 1.617\ell$,

$|Ba| = 0.618\ell$. Thus, $g \geq 0.38\ell$. Therefore, the gap ratio just after point insertion on the pentagon is at most

$$\frac{G}{g} \cdot 1.34 = 5.99.$$

– Then we consider point insertion in an isosceles acute triangle. Since all such isosceles acute triangles are of the same shape, after insertion, the gap ratio will be the same too. This case can be analyzed as shown in Figure 4(a), i.e., by considering the insertion in the acute triangle $AJj$. Due to the selection criteria, the maximal gap is the spherical diameter. Thus,

$$G = \frac{2 \cdot |AJ| \cdot |Jj| \cdot |jA|}{\sqrt{(|AJ| + |Jj| + |jA|)(-|AJ| + |Jj| + |jA|)(|AJ| - |Jj| + |jA|)(|AJ| + |Jj| - |jA|)}}.$$

Since $|AJ| = |jA| = \ell$ and $|Jj| = \sqrt{|AJ|^2 + |jA|^2 - 2 \cdot |AJ| \cdot |jA| \cos \angle JAj} = 0.618\ell$, we have $G = 1.05\ell$.

After insertion, the minimal gap is the distance between $p_1$ and $j$, Thus,

$$g = \sqrt{|Jj|^2 + |Jp_1|^2 - 2 \cdot |Jj| \cdot |Jp_1| \cos \angle jJp_1}.$$

Note that the triangle $Jjp_1$ is still an isosceles acute triangle with the angle $\angle jJp_1 = \pi/5$, and we have $g = 0.382\ell$. Therefore, the gap ratio is at most

$$\frac{G}{g} \cdot 1.34 = 3.68.$$

– Lastly, we consider point insertion in an isosceles obtuse triangle. Similar to the previous case, we only need to consider the insertion on the triangle $AeE$, which is shown in Figure 4(b). Since it is an isosceles obtuse triangle, the maximal gap is at most twice the distance $|eh_1|$. After insertion, the triangle is partitioned into an isosceles obtuse triangle $Aeh_1$ and an isosceles acute triangle $Ah_1E$. In this case, the minimal gap is the distance between $e$ and $h_1$, i.e., $|eh_1|$. Thus, the gap ratio at this stage is at most

$$\frac{G}{g} \cdot 1.34 \leq 2.68.$$

Combining all the above cases, we have the following concluding theorem.

**Theorem 1.** *The maximal gap ratio of the insertion strategy is at most 5.99.*

## 3 Conclusion and Discussion

Uniform insertion of points is an interesting problem in computer science. With the help of the dodecahedron and the pentagon, we give a two-phase insertion strategy with gap ratio of no more than 5.99 in the paper.

Is it possible to further reduce the gap ratio by using other structures? How about some regular simpler structure, e.g., isocahedron? If we split the isocahedron into four congruent sub-triangles regularly, the gap ratio will be larger since the newly inserted points are on the side of the isocahedron. From the definition, the maximal gap is the spherical diameter of the largest empty circle while the minimal gap is the minimal spherical distance between two inserted points. So, if points are inserted on the side of some configuration, the ratio might be not good.

## Acknowledgements

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## References

1. T. Asano, Online uniformity of integer points on a line. Information Processing Letters 109(2008), pp. 57-60.
2. T. Asano, S. Teramoto, On-line uniformity of points. In: Book of Abstracts for 8th Hellenic-European Conference on Computer Mathematics and its Applications, Athens, Greece, September, 2007, pp. 21-22.
3. A. Badanidiyuru, R. Kleinberg, Y. Singer, Analytically exact spiral scheme for generating uniformly distributed points on the unit sphere. J. Comput. Science 2(1): 88-91 (2011)
4. A. Bishnu, S. Desai, A. Ghosh, M. Goswami, S. Paul, Uniformity of Point Samples in Metric Spaces Using Gap Ratio In Proc. of TAMC 2015, pp. 347-358.
5. B. Chazelle, The Discrepancy Method: Randomness and Complexity. Cambridge University Press, 2000.
6. J.M. Cook, An Efficient Method for Generating Uniformly Distributed Points on the Surface on an $n$-Dimensional Sphere (Remarks). Commun. ACM 2(10): 26 (1959).
7. C.R. Collins, and K. Stephenson, A circle packing algorithm. Computational Geometry: Theory and Applications, 25(3), 2003, pp. 233-256.
8. J.S. Hicks, R.F. Wheeling, An Efficient Method for Generating Uniformly Distributed Points on the Surface of an $n$-Dimensional Sphere. Commun. ACM 2(4): 17-19 (1959)
9. C.G. Koay, Distributing points uniformly on the unit sphere under a mirror reflection symmetry constraint. J. Comput. Science 5(5): 696-700 (2014)
10. J. Matoušek, Geometric Discrepancy. Springer, 1991.
11. K.J. Nurmela, and P.R.J. Östergård, More Optimal Packings of Equal Circles in a Square. Discrete & Computational Geometry 22(3): 439-457 (1999)
12. S. Teramoto, T. Asano, N. Katoh, B. Doerr, Inserting points uniformly at every instance. IEICE Trans. Inf. Syst. E89-D (8) (2006) 2348-2356.

13. J. J. Thomson, On the Structure of the Atom: an Investigation of the Stability and Periods of Oscillation of a number of Corpuscles arranged at equal intervals around the Circumference of a Circle; with Application of the Results to the Theory of Atomic Structure, Philosophical Magazine Series 6, Volume 7, Number 39, pp. 237-265, March 1904.
14. Y. Zhang, Z. Chang, F.Y.L. Chin, H-F Ting, and Y.H. Tsin, Uniformly Inserting Points on Square Grid. Information Processing Letters, 111(2011) 773-779.