Probability Review

Random Variable

Random Variables

In many cases, we associate a *numeric value* with each outcome of an experiment.

For instance,

consider the experiment of flipping a coin 10 times, each outcome can be associated with

- the number of heads,
- the difference between heads & tails, etc.

Each quantity above is called a ***random variable***; note that its value depends on the outcome of the experiment and is not fixed in advance.

Formally speaking

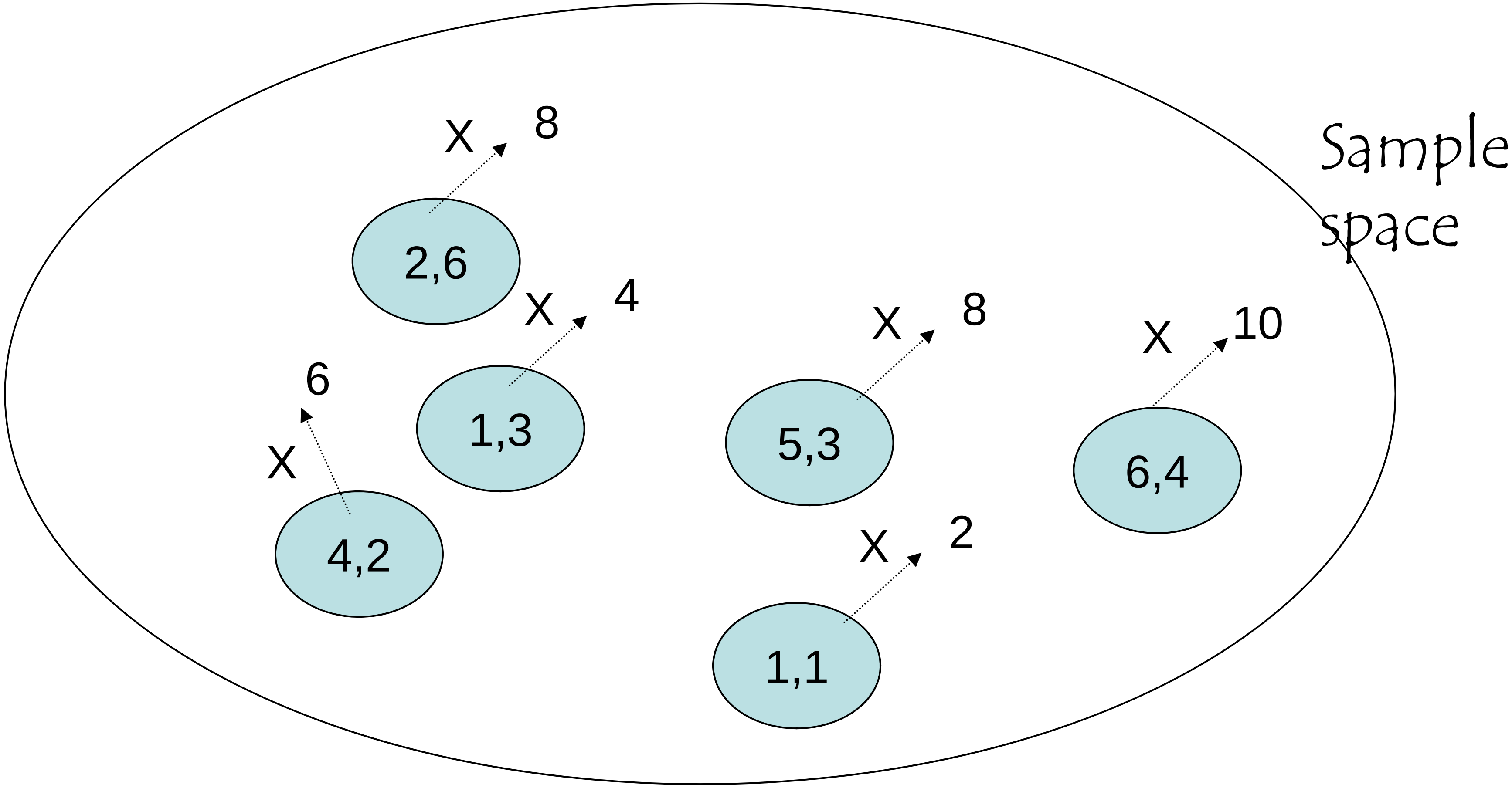
With respect to an experiment,

a random variable is a ***function***

- from the set of possible outcomes Ω
- to the set of real numbers.

NB. A random variable is characterised by the sample space of an experiment.

Example: Let X be the sum of the numbers obtained by rolling a pair of fair dice.
There are 36 possible outcomes, each defines a value of X (in the range from 2 to 12).



Random variables and events

A more intuitive way to look at the random variable X is to examine the probability of each possible value of X.

E.g., consider the previous example:

- Let p($X=3$) be the *probability of the event* that the sum of the two dice is 3.

  This event comprises two outcomes, (1,2) and (2,1).
- p($X=3$) = 2/36.

Random variables and events

In general, for any random variable ***X***,
p(***X*** $= i$) = the sum of the probability of all the outcomes *y* such that ***X***(*y*) = i.

$\sum_{i \in \text{the range of } X} p(X = i) = 1$

Expected value

In the previous example, what is the expected value (average value) of X?

Out of the 36 outcomes,

(1,1): X=2

(1,2), (2,1): X=3

(1,3), (2,2), (3,1): X=4

(1,4), (2,3), (3,2), (4,1): X=5

(1,5), (2,4), (3,3), (4,2), (5,1): X=6

(1,6), (2,5), (3,4), (4,3), (5,2), (6,1): X=7

(2,6), (3,5), (4,4), (5,3), (6,2): X=8

(3,6), (4,5), (5,4), (6,3): X=9

(4,6), (5,5), (6,4): X=10

(5,6), (6,5): X=11

(6,6): X=12

Expected value of X =

(2 + 3x2 + 4x3 + 5x4
+ 6x5 + 7x6 + 8x5 + 9x4
+ 10x3 + 11x2 + 12) / 36

= 7

Definition

Consider an experiment with a sample space **S**. For any outcome y in **S**, let p(y) be the probability y occurs.

X: S → Z

Let X be an integer random variable over **S**. That is, every outcome y in S defines a value of X, denoted by X(y).

We define the expected value of X to be

$$\sum_{y \in \mathbf{S}} p(y)\, X(y)$$

or equivalently,

$$\sum_{i \in Z} p(X = i)\, i$$

Example

What is the *expected number of heads* in flipping a fair coin four times?

Prob [# of heads = 1]

$1 \times C(4,1)\ 1/2\ (1/2)^3$

$+\ 2 \times C(4,2)\ (1/2)^2\ (1/2)^2$

$+\ 3 \times C(4,3)\ (1/2)^3\ 1/2$

$+\ 4 \times (1/2)^4$

$= 4/16 + 2 \times 6/16 + 3 \times 4/16 + 4 \times 1/16$

$= 2$

Example: Network Protocol

Repeat

flip a fair coin twice;

if “head + head” then send a packet to the network;

until “sent”

What is the expected number of iterations used by the protocol?

Example

Let p be the probability of success within each trial.
Let q be the probability of failure within each trial.

$\sum_{i=1 \text{ to } \infty}$ (prob of sending the packet in the i-th trial) i

$= \sum_{i=1 \text{ to } \infty} (q^{i-1}p)\, i$

$= p \sum_{i=1 \text{ to } \infty} (q^{i-1})\, i$

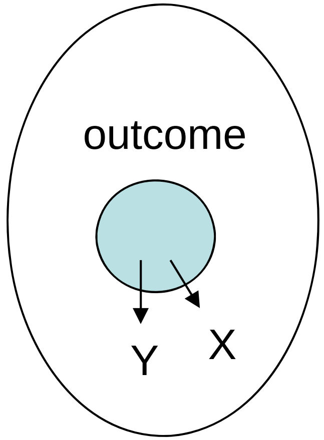
$= p \,/\, (1 - q)^2$

$= 1 \,/\, p$

Useful rules for deriving expected values

Let X and Y be random variables on a space S, then

- X + Y is also a random variable on S, and
- E(X + Y) = E(X) + E(Y).



Proof.

E(X + Y) $= \sum_{t \text{ in } S} p(t) \, (X(t) + Y(t))$

$= \sum_{t \text{ in } S} p(t) \, (X(t)) \; + \; \sum_{t \text{ in } S} p(t) \, Y(t)$

= E(X) + E(Y)

Example

- Use the "sum rule" to derive the expected value of the sum of the numbers when we roll a pair of fair dice (denoted by X).
- Suppose the dice are colored red & blue.
- Let X_1 be the number on the red dice when we roll a pair of red & blue dice, and similarly X_2 for the blue dice.

  $E(X_1) = E(X_2) = ?$
- Obviously, $X = X_1 + X_2$. Thus, $E(X) = E(X_1) + E(X_2) =$.

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What about product?

Let X and Y be two random variables of a space S.

Is $E(XY) = E(X) E(Y)$?

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What about product?

Let X and Y be two random variables of a space S. Is E(XY) = E(X) E(Y)?

Example 1: Consider tossing a coin twice. Associate “head” with 2 and “tail” with 1.

What is the expected value of the product of the numbers obtained in tossing a coin twice.

- (1,1) $\rightarrow$ 1; (1,2) $\rightarrow$ 2; (2,1) $\rightarrow$ 2; (2,2) $\rightarrow$ 4
- Expected product = (1+2+2+4) / 4 = 2.25
- Expected value of 1st flip: (1+2)/2 = 1.5
- Expected value of 2nd flip: (1+2)/2 = 1.5

Note that 2.25 = 1.5 x 1.5!

Counter Example

Consider the previous experiment again. Define a random variable X as follows:

X=(the first number) x (the sum of the two numbers)

- (1,1) -> 1 x 2 = 2
- (1,2) -> 1 x 3 = 3
- (2,1) -> 2 x 3 = 6
- (2,2) -> 2 x 4 = 8
- Expected value of 1st number = 1.5
- Expected sum = (2 + 3 + 3 + 4)/4 = 3

Expected value = 4.75

$4.75 \neq 1.5 \times 3$

Why! Because the first # & the sum are not independent.

Counter Example

Consider the experiment of flipping a fair coin twice.

Expected (number of) heads = 1

Expected tails = 1

Expected heads × expected tails = 1

Expected (heads × tails) = 2 / 4 = 0.5

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Independent random variables

Two random variables X and Y over a sample space S are independent if

for **all** real numbers r_1 and r_2,

the events "**X** = r_1" and "**Y** = r_2" are independent,

i.e., $p(X = r_1 \text{ and } Y = r_2) = p(X = r_1) \times p(Y = r_2)$.

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Product rule.

If X and Y are independent random variables on a space S, then $E(XY) = E(X)\,E(Y)$.