COMP8601: Advanced Topics in Theoretical Computer Science
Lecture 8: ϵ-Nets, VC-dimension
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1 ϵ-Net

Suppose X is a set with some distribution D, and C is a class of boolean functions, each of which has the form $f : X \to \{0, 1\}$. We can think of each function f as a concept, labelling each point in X as positive (1) or negative (0). The goal is to obtain a small subset $S \subset X$ such that for each function $f \in C$, if a large fraction (weighted according to distribution D) of points in X are marked as positive under f, then there exists at least one point in S that is also marked positive under f. We use $E_X[f] := E_{x \in D(X)}[f(x)]$ to denote the expectation of $f(x)$, where x is a point drawn from X with distribution D.

Definition 1.1 (ϵ-Net) *An ϵ-net S for a set X with distribution D under a class C of boolean functions on X is a subset satisfying the following:*

For each $f \in C$, if $E_X[f] \geq \epsilon$, then there exists $x \in S$ such that $f(x) = 1$.

Trivially, we could take $S := X$ as an ϵ-net. However, we would want the cardinality of S to be small, even though X or C might be infinite.

We assume that we are able to sample points independently from X under distribution D. The straightforward way to construct a net is to sample an enough number of points.

For $0 < \epsilon \leq 1$, we define $C_\epsilon := \{f \in C : E_X[f] \geq \epsilon\}$.

Example

Suppose X are points in the plane $\mathbb{R}^2$ with some distribution, and C is the class of functions, each of which corresponds to an axis-aligned rectangle that marks the points inside 1 and 0 otherwise. We would later see that for every $0 < \epsilon \leq 1$, there is some finite sized ϵ-net S_ϵ, i.e., if a rectangle contains more than ϵ (weighted) fraction of points in X, then it must contain a point in S_ϵ.

1.1 Simple Case: When C is finite

Theorem 1.2 *Suppose C is finite and S is a subset obtained by sampling from X independently m times. (There could be repeats, and so S could have size smaller than m.) If $m \geq \frac{1}{\epsilon}(\ln |C| + \ln \frac{1}{\delta})$, then with probability at least $1 - \delta$, S is an ϵ-net.*

Proof: Observe that S is an ϵ-net, if for all $f \in C_\epsilon$, there is some point $x \in S$ such that $f(x) = 1$. Fix any $f \in C_\epsilon$, the probability that a point sampled from X would be labelled 1 is at least ϵ. Hence, the failure probability that all points in S are labelled 0 under f is at most $(1-\epsilon)^m \leq e^{-\epsilon m}$.

Using union bound, the probability that the set S fails for some $f \in C_\epsilon$ is at most $|C_\epsilon| e^{-\epsilon m} \leq |C| e^{-\epsilon m}$, which is at most δ, when $m \geq \frac{1}{\epsilon}(\ln |C| + \ln \frac{1}{\delta})$. ■

1.2 Extending to Infinite C

Observe that for a fixed subset S in X, if two functions f and f' agree on every point in S, then essentially they are the same from the viewpoint of S. Hence, for every fixed set S of size m, there are effectively only 2^m boolean functions. However, there are still some issues.

1. There are still too many functions. Recall in the proof, we used the union bound to analyze the failure probability $|C| \cdot e^{-\epsilon m} \leq 2^m \cdot e^{-\epsilon m}$. However, this is not useful as the last quantity is larger than 1.

2. After we fix some S, there is no more randomness. Hence, we cannot even argue that the probability that S is bad for even one f is at most $(1-\epsilon)^m$.

For the first issue, we would add more assumptions to the class C of functions to obtain a better guarantee. The second issue is technical and can be resolved by using the technique of conditional probability and expectation.

2 VC-Dimension: Limiting the Number of Boolean Functions on a Subset

Definition 2.1 *Given a set X and a class C of boolean function on X, a subset $S \subseteq X$ is said to be shattered by C, if for all subsets U of S, there exists $f \in C$ such that for all $x \in U$, $f(x) = 1$ and for all $x \in S \setminus U$, $f(x) = 0$.*

The VC-dimension of (X, C) is the maximum cardinality of a subset $S \subseteq X$ that is shattered by C. In other words, the VC-dimension of (X, C) is at least d if there exists $S \subseteq X$, where $|S| = d$, such that S is shattered by C.

Example. Consider $X = \mathbb{R}^2$ and C is the class where each function corresponds to an axis-aligned rectangle that labels each points inside it 1 and otherwise 0. Observer that $S = \{(1,0), (-1,0), (0,1), (0,-1)\}$ can be shattered by C. However, one can show that no 5 points on the plane can be shattered by C.

Definition 2.2 *Suppose $S \subseteq X$ and $f : X \to \{0,1\}$. Then, the projection of f on S is the boolean function $f \mid_S : S \to \{0,1\}$ such that for all $x \in S$, $f \mid_S (x) = f(x)$. Given a class C of boolean functions, the projection $C(S)$ of C on S is the class $C(S) := \{f \mid_S : f \in C\}$.*

Given non-negative integers m and d, we denote $\binom{m}{\leq d} := \sum_{i=0}^{d} \binom{m}{i}$.

Theorem 2.3 (Sauer's Lemma [1]) *Suppose C is a class of boolean functions on X and the VC-dimension of (X, C) is at most d. Let S be a subset of X of size m. Then, the cardinality of the projection $C(S)$ is at most $\binom{m}{\leq d}$. In particular, when $m \geq d \geq 1$, this is at most $(\frac{em}{d})^d$.*

Proof: We prove by induction on d and m. For the base cases $d = 0$ or $m = 1$, we leave it to the readers to verify the claim. Suppose we have S, where $|S| = m > 1$, and the VC-dimension of

(X, C) is $d \geq 1$. We give an upper bound on $|C(S)|$.

Let $x \in S$ and define $S' := S \setminus \{x\}$. Define $C(S')^\dagger \subseteq C(S')$ to be the set of functions f in $C(S')$ such that there exists $f_1, f_2 \in C(S)$, where f_1 and f_2 disagree on x and $f_1 \mid_{S'} = f_2 \mid_{S'} = f$.

Consider the projection of C on S'. It follows that each function in $C(S')^\dagger$ can be viewed as a "merge" of 2 functions in $C(S')$. Hence, it follows that $|C(S)| = |C(S')| + |C(S')^\dagger|$.

By induction hypothesis, we immediately have $|C(S')| \leq \binom{m-1}{\leq d}$.

We next show that the VC-dimension of $(S', C(S')^\dagger) \leq d - 1$. Suppose $C(S')^\dagger$ shatters a subset $U \subseteq S'$. Then, it follows immediately that $C(S)$ shatters $U \cup \{x\}$, which has size at most d, since the VC-dimension of (X, C) is at most d. It follows $|U| \leq d - 1$. Hence, by induction hypothesis $|C(S')^\dagger| \leq \binom{m-1}{\leq d-1}$.

By observing that $\binom{m}{i} = \binom{m-1}{i} + \binom{m-1}{i-1}$, we conclude that $|C(S)| \leq \binom{m-1}{\leq d} + \binom{m-1}{\leq d-1} = \binom{m}{\leq d}$.

Since $0 < \frac{d}{m} \leq 1$, we have $|C| \leq \sum_{i=0}^{d} \binom{m}{i} \leq \left(\frac{m}{d}\right)^d \sum_{i=0}^{d} \binom{m}{i} \left(\frac{d}{m}\right)^i \leq \left(\frac{m}{d}\right)^d \sum_{i=0}^{m} \binom{m}{i} \left(\frac{d}{m}\right)^i = \left(\frac{m}{d}\right)^d \left(1 + \frac{d}{m}\right)^m \leq \left(\frac{em}{d}\right)^d$, where the last inequality follows from $1 + x \leq e^x$ for $x \in \mathbb{R}$. ■

Here is the result relating VC-dimension of (X, C) and the number of independent samples that is sufficient to form an ϵ-net for X under C.

Theorem 2.4 (Number of Samples for Class with Bounded VC-Dimension) *Suppose (X, C) has VC-dimension at most d. Then, suppose S is a subset obtained by sampling from X independently m times (and removing repeated points). If $m \geq \max\{\frac{4}{\epsilon} \log \frac{2}{\delta}, \frac{8d}{\epsilon} \log \frac{16e}{\epsilon}\}$, then with probability at least $1 - \delta$, S is an ϵ-net.*

Intuition. Observe that $|C(S)| \leq \binom{m}{\leq d} \leq (\frac{em}{d})^d$, for $m \geq d \geq 2$. Hence, if we use the "bogus" union bound, the failure probability would be at most $|C(S)| \cdot e^{-\epsilon m} \leq (\frac{em}{d})^d \cdot e^{-\epsilon m}$. When m is large enough as specified, this quantity is less than δ.

3 Conditional Probability and Expectation as Random Variables

We see that if (X, C) has VC-dimension d, then the projection of C on some subset $S \subseteq X$ with $|S| = m$ has size $|C(S)| \leq (\frac{em}{d})^d$. When we sample a subset S, we would like to analyze the size of $C(S)$, conditioned on the fact that S is sampled. We need some formal notation to analyze this.

Definition 3.1 (Random Object) *Suppose $\mathcal{P} = (\Omega, \mathcal{F}, Pr)$ is a probability space. A random object W taking values in some set $\mathcal{U}$ is a function $W : \Omega \to \mathcal{U}$. For $u \in \mathcal{U}$, $\{W = u\}$ is the event $\{\omega \in \Omega : W(\omega) = u\}$.*

Example.

1. A $\{0, 1\}$-random variable is a special case when $\mathcal{U} = \{0, 1\}$.

2. Suppose we flip a fair coin repeatedly, and W is the outcome of the first 2 flips. In this case, $\mathcal{U} = \{H, T\}^2$.

Definition 3.2 (Conditional Probability as a Random Variable) *Suppose $\mathcal{P} = (\Omega, \mathcal{F}, Pr)$*

is a probability space, and $A \in \mathcal{F}$ is an event. Let $W : \Omega \to \mathcal{U}$ be a random object. Then, the conditional probability $Pr[A|W]$ can be interpreted in two ways:

1. *$Pr[A|W] : \mathcal{U} \to [0, 1]$ is a function such that for $u \in \mathcal{U}$, $Pr[A|W](u) := Pr[A|W = u]$.*
2. *$Pr[A|W] : \Omega \to [0, 1]$ is a random variable defined by $Pr[A|W](\omega) := Pr[A|W_\omega]$, where $W_\omega := \{\omega' \in \Omega : W(\omega') = W(\omega)\}$ is the event that W equals to $W(\omega) \in \mathcal{U}$.*

Definition 3.3 (Conditional Expectation as a Random Variable) *Suppose $\mathcal{P} = (\Omega, \mathcal{F}, Pr)$ is a probability space, and $Y : \Omega \to \mathbb{R}$ is a random variable. Let $W : \Omega \to \mathcal{U}$ be a random object. Then, the conditional expectation $E[Y|W]$ can be interpreted in two ways:*

1. *$E[Y|W] : \mathcal{U} \to \mathbb{R}$ is a function such that for $u \in \mathcal{U}$, $E[Y|W](u) := E[Y|W = u]$.*
2. *$E[Y|W] : \Omega \to \mathbb{R}$ is a random variable defined by $E[Y|W](\omega) := E[Y|W_\omega]$, where $W_\omega := \{\omega' \in \Omega : W(\omega') = W(\omega)\}$ is the event that W equals to $W(\omega) \in \mathcal{U}$.*

Since the conditional probability $Pr[A|W]$ and the conditional expectation $E[Y|W]$ are random variables themselves, we can take expectation of them.

Fact 3.4 *Let the event A, the random variable Y and the random object W be defined as above. Then, $E[Pr[A|W]] = Pr[A]$ and $E[E[Y|W]] = E[Y]$.*

Example. Consider the probability space associated with flipping a fair coin repeatedly. Let W be the outcome of the first 2 flips, and Y be the number of flips that a head first appears. As before, we have $\mathcal{U} = \{H, T\}^2$. Consider the conditional expectation $E[Y|W]$.

1. We have $E[Y|W = \{H, H\}] = 1$, $E[Y|W = \{H, T\}] = 1$, $E[Y|W = \{T, H\}] = 2$. Finally, $E[Y|\{T, T\}] = 2 + E[Y] = 4$.
2. Hence, $E[E[Y|W]] = \frac{1}{4}(1 + 1 + 2 + 4) = 2 = E[Y]$.

3.1 Using Conditional Probability to Bound Failure Probability

Recall that we are drawing independent samples from X to form a subset S of size m in the hope that S would be an ϵ-net for the class C of functions. Suppose further that (X, C) has VC-dimension d.

Let A be the event that S is not an ϵ-net under C. In particular, let A_f be the event that for all $x \in S$, $f(x) = 0$. Recall that $C_\epsilon := \{f \in C : E_X[f] \geq \epsilon\}$. We wish to find a good upper bound for $Pr[A] = Pr[\cup_{f \in C_\epsilon} A_f]$.

Using conditional probability, we have $Pr[A] = E[Pr[A|S]]$. Observe that if we fix S, then the set S fails for the function $f \in C$ *iff* S fails for $f' := f\,|_S \in C(S)$. Hence, $Pr[A|S] = Pr[\cup_{f \in C_\epsilon} A_f|S] = Pr[\cup_{f' \in C_\epsilon(S)} A_{f'}|S] \leq \sum_{f' \in C_\epsilon(S)} Pr[A_{f'}|S]$.

Observe that the summation contains at most $|C_\epsilon(S)| \leq |C(S)| \leq (\frac{em}{d})^d$ terms. Hence, it suffices to give a good upper bound on $p^* := \max_{f' \in C_\epsilon(S)} Pr[A_{f'}|S]$. However, as we mention before, if we condition on S, there is no more randomness, since $Pr[A_f|S]$ is either 0 or 1. Hence, we can have $p^* = 1$. We shall see next time how we can resolve this by introducing extra randomness in the analysis.

4 Homework Preview

1. **VC-dimension of Axis-aligned rectangles.**
   (a) Prove that no 5 points on the plane $\mathbb{R}^2$ can be shattered by the class C of axis-aligned rectangles (that map points inside a rectangle 1 and otherwise 0).
   (b) Compute the VC-dimension of the class C_k of k-dimensional axis-aligned rectangles in $\mathbb{R}^k$. In particular, you need to find a number d such that there exist d points in $\mathbb{R}^k$ that can be shattered by the C_k, and prove that any $d+1$ points in $\mathbb{R}^k$ cannot be shattered by C_k.
2. **Conditional Expectation.** Suppose $Y : \Omega \to \mathbb{R}$ is a random variable and $W : \Omega \to \mathcal{U}$ is a random object defined on the same probability space $(\Omega, \mathcal{F}, Pr)$. Prove that $E[Y] = E[E[Y|W]]$. You may assume that both Ω and $\mathcal{U}$ are finite.

References

[1] N Sauer. On the density of families of sets. *Journal of Combinatorial Theory, Series A*, 13(1):145 – 147, 1972.