

Online Submodular Maximization with Free Disposal: Randomization Beats $\frac{1}{4}$ for Partition Matroids*

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Abstract

We study the online submodular maximization problem with free disposal under a matroid constraint. Elements from some ground set arrive one by one in rounds, and the algorithm maintains a feasible set that is independent in the underlying matroid. In each round when a new element arrives, the algorithm may accept the new element into its feasible set and possibly remove elements from it, provided that the resulting set is still independent. The goal is to maximize the value of the final feasible set under some monotone submodular function, to which the algorithm has oracle access.

For k -uniform matroids, we give a deterministic algorithm with competitive ratio at least 0.2959, and the ratio approaches $\frac{1}{\alpha_\infty} \approx 0.3178$ as k approaches infinity, improving the previous best ratio of 0.25 by Chakrabarti and Kale (IPCO 2014), Buchbinder et al. (SODA 2015) and Chekuri et al. (ICALP 2015). We also show that our algorithm is optimal among a class of deterministic monotone algorithms that accept a new arriving element only if the objective is strictly increased.

Further, we prove that no deterministic monotone algorithm can be strictly better than 0.25-competitive even for partition matroids, the most modest generalization of k -uniform matroids, matching the competitive ratio by Chakrabarti and Kale (IPCO 2014) and Chekuri et al. (ICALP 2015). Interestingly, we show that randomized algorithms are strictly more powerful by giving a (non-monotone) randomized algorithm for partition matroids with ratio $\frac{1}{\alpha_\infty} \approx 0.3178$.

Finally, our techniques can be extended to a more general problem that generalizes both the online submodular maximization problem and the online bipartite matching problem with free disposal. Using the techniques developed in this paper, we give constant-

competitive algorithms for the submodular online bipartite matching problem.

1 Introduction

We study online submodular maximization with free disposal under a matroid constraint. Let Ω be the ground set of elements, $f : 2^\Omega \rightarrow \mathbb{R}_+$ be a non-negative submodular objective function, and $\mathcal{I} \subseteq 2^\Omega$ be a collection of feasible subsets in Ω that the algorithm can choose from. The goal is to find $S \in \mathcal{I}$ such that $f(S)$ is maximized. In this paper, we focus on the case when \mathcal{I} forms a matroid, i.e., a set of elements S is feasible if it is independent with respect to the matroid.

The offline version [51, 52] has been extensively studied due to its many applications, such as the maximum coverage problem with group budget constraints [16, 45], the separable assignment problem [30, 31, 12], the assignment learning problem [36, 37], the sequence optimization problem [20], and the submodular welfare maximization problem [54, 24, 21].

In the online version (without free disposal), the elements in Ω arrive in rounds in an arbitrary order. The algorithm maintains a feasible set $S \in \mathcal{I}$, which is initially empty. In each round, the algorithm must irrevocably decide whether to add the arriving element u into S (provided that $S + u \in \mathcal{I}$) without knowing the future elements. We assume the algorithm has value oracle access to function f on any subset of elements arrived so far. However, this version of the problem has no non-trivial competitive ratio even for the simple constraint $|S| \leq 1$.¹

Buchbinder et al. [11] explicitly considered the online version with *free disposal*². In this model, in each round, the algorithm may also remove elements from its current feasible set S , as well as adding the new

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¹Consider a sequence of elements with value $1, \rho, \rho^2, \dots, \rho^n$ for some $\rho \gg 1$ and n unknown to the algorithm.

²The terms free disposal [25] and preemption [11] have both been used in the literature. We will use free disposal throughout this paper.

arriving element into S , as long as the resulting S is still in \mathcal{I} . (However, elements that have not been chosen at their arrival, or have been disposed of cannot be retrieved back.) They pointed out that a result by Chakrabarti and Kale [13] implies a 0.25-competitive algorithm for maximizing monotone submodular functions online under a k -uniform matroid constraint, i.e., for some positive integer k , \mathcal{I} consists of all subsets with cardinalities at most k . Buchbinder et al. [11] also proposed a different 0.25-competitive algorithm which leads to a 0.0893-competitive randomized algorithm for non-monotone submodular functions under a k -uniform matroid constraint. They also showed several hardness results for various settings.

Streaming Model. Chakrabarti and Kale [13] and Chekuri et al. [15] considered streaming version of this problem in which the algorithm has limited memory. They consider even more general independent systems than matroids, and their algorithms for the case of matroids can be interpreted as an online algorithm with free disposal that is 0.25-competitive. We summarize the previous results in Table 1.

Matroid	Objective	Algorithm	Hardness
k -Uniform	Monotone	0.25 [11, 13, 15]	0.5 [11]
	General	0.0893 [11]	0.438 [11]
General	Monotone	0.25 [13, 15]	0.5 [11]
	General	-	0.438 [11]

Table 1: A summary of previous results

Our Contributions. We make contribution to the problem by improving both the upper and lower bounds on the competitive ratios in various settings. A summary is given in Table 2.

Monotone Algorithm. A deterministic algorithm is *monotone* if, after each round, it either keeps the same the set of chosen elements, or makes changes that strictly increase the objective (see the precise Definition 2.1).

Why monotone algorithms? First of all, monotonicity of algorithms is a natural requirement for some applications. Consider the example of managing a soccer team proposed by Buchbinder et al. [11]. It would be difficult to talk the board and the fan base into a transfer of players without immediate benefits to the team. Further, to our knowledge, all known algorithms in the literature for monotone submodular objectives are deterministic and monotone. Hence, it would be interesting to fully understand the potential of this family of algorithms.

Our first contribution is an improved algorithm

for the case of k -uniform matroids. We propose a deterministic monotone algorithm (Section 3) that is at least 0.2959-competitive for monotone submodular functions, improving the previous ratio of 0.25 [11, 13, 15]. As k tends to infinity, our competitive ratio approaches $\frac{1}{\alpha_\infty} \approx 0.3178$ (from below), where α_∞ is the unique root of $\alpha = e^{\alpha-2}$ that is greater than 1. Further, we obtain a matching hardness result (Section 5) in the sense that for any $\epsilon > 0$, there is some sufficiently large $k > 0$ such that no deterministic monotone algorithm has competitive ratio at least $\frac{1}{\alpha_\infty} + \epsilon$ under a k -uniform matroid constraint.

For general matroids, we show that no deterministic monotone algorithm is strictly better than 0.25-competitive even for partition matroids, the most modest generalization of k -uniform matroids (Section 6). Our hardness result matches the competitive ratio by [13, 15].

Randomized Algorithms on Partition Matroids. Given the hardness for deterministic monotone algorithms, it is natural to ask whether we could get a better competitive ratio using randomized (and non-monotone) algorithms. We consider the setting that the adversary first fixes the arrival order before the algorithm samples its randomness. We give affirmative answer to this question for the case of partition matroids. While a partition matroid can be viewed as a union of disjoint uniform matroids, our k -uniform algorithm fails to generalize directly due to the case of a union of 1-uniform matroids. We handle a single 1-uniform matroid using the trivial algorithm that keeps the most valuable element, but this trick no longer works when there is a union of many 1-uniform matroids.

Our high-level idea is to use randomized algorithms to effectively allow picking only a fraction of each element and, thus, treating each partition as effectively having large size (with respect to tiny fractions of the elements). There are some technical obstacles. First of all, any rounding scheme that does not incur an intrinsic loss in the objective, e.g., pipage rounding [1, 33], fails to work in the online setting. As a result, we settle for an online rounding scheme that loses a $1 - \frac{1}{e}$ factor in the objective in the worst case. However, due to the intrinsic loss from rounding, a naïve competitive analysis gives only the product of $\frac{1}{\alpha_\infty}$ and $1 - \frac{1}{e}$ which is smaller than $\frac{1}{4}$. We avoid losing an extra $1 - \frac{1}{e}$ factor observing that the scenario that gives rise to a $\frac{1}{\alpha_\infty}$ ratio for the fractional algorithm and the scenario that incurs a rounding loss of $1 - \frac{1}{e}$ do not occur simultaneously. To instantiate this observation, we introduce a novel inequality (Lemma 4.5) that allows us to directly compare the optimal objective and the expected value of f for the fractional solution after the

Matroid	Algorithm		Hardness for Det. Alg.		
	Det. Alg.		Rand. Alg.	General Alg.	Monotone Alg.
k -Uniform	Worst k	$k \rightarrow \infty$	$\frac{1}{\alpha_\infty} \approx 0.318$	0.5 [11]	$\frac{1}{\alpha_\infty} \approx 0.318$ (Thm. 5.1)
	0.296 (Thm. 3.1)	$\frac{1}{\alpha_\infty} \approx 0.318$			
Partition	0.25 [11, 13, 15]		(Thm. 4.1)	0.382 (Thm. 6.2)	0.25 (Thm. 6.1)

Table 2: A summary of the main results in this paper. The objective is monotone. The deterministic algorithms are monotone. The hardness results are with respect to deterministic algorithms.

rounding.

Dichotomy between Deterministic and Randomized Algorithms. Our improved competitive ratio for partition matroids shows that (non-monotone) randomized algorithms are strictly more powerful, as our randomized algorithm on partition matroids has ratio $\frac{1}{\alpha_\infty} \approx 0.3178$, which is achieved by our “continuous” algorithm. Conventional discrete algorithms can approach this ratio arbitrarily closely.

Extensions. Using the new insights we get for monotone objectives, we further introduce a randomized algorithm that is 0.1145-competitive for non-monotone objectives under uniform matroid constraints (where details can be found in the full version[14]), improving the previous 0.0893 ratio [11].

Generalized Online Bipartite Matching. Our techniques in fact solve a more general problem that generalizes both the online submodular maximization problem and the online bipartite matching problem with free disposal that was first proposed in [25]. In this submodular online bipartite matching problem, each offline node corresponds to some agent $\lambda \in \Lambda$, and each online node corresponds to an item $u \in \Omega$. Each agent λ has an evaluation function $f^\lambda : 2^\Omega \rightarrow \mathbb{R}_+$, and is also associated with a matroid $(\Omega, \mathcal{I}^\lambda)$.

The online submodular maximization problem is a special case with only one agent, and the online bipartite matching problem is a special case when each agent is under a 1-uniform matroid constraint. We show that our $\frac{1}{\alpha}$ -competitive deterministic online algorithm for a single offline node that is defined in Section 3 (and one for general matroid defined in the full version[14]) induce $\frac{1}{\alpha+1}$ -competitive algorithms for submodular online bipartite matching (where details are in the full version[14]).

Streaming Model. In contrast to previous approaches [11, 13, 15], our improved algorithm (Algorithm 1) for uniform matroid cannot be fitted into the streaming model. As we shall see, it is crucial for the algorithm to remember all the items that have been selected, where the limited space is insufficient. This might represent a separation between the streaming and the online version

of the model.

Paper Organization. The monotone algorithm for a k -uniform matroid is given in Section 3. The randomized algorithm for a partition matroid is given in Section 4. The hardness results for uniform matroids and general matroids are given in Section 5 and Section 6, respectively. The details for the online algorithm of the submodular online bipartite matching problem, and a randomized algorithms for non-monotone objective functions are supplied in the full version[14]. We also reprove the $\frac{1}{4}$ competitive ratio under a general matroid for completeness in the full version[14], which is useful for the submodular online bipartite matching problem.

Other Related Work. We have already discussed the related work on online submodular maximization. There is a vast literature on submodular maximization in different settings. We will review some of the results that are most related.

In the offline setting, Buchbinder et al. introduced a 0.5-approximate randomized algorithm for maximizing a non-monotone submodular function with no constraint [10]. Feige et al. had previously proved that 0.5 is the best possible for this setting [23]. Recently, Buchbinder and Feldman [9] obtained a deterministic algorithm which also achieves the optimal 0.5 ratio. For a uniform matroid constraint, Nemhauser et al. [51] showed a $(1 - \frac{1}{e})$ -approximate algorithm for monotone objectives, which is optimal [50]. Feige [22] further proved that even when the objective is a coverage function, no algorithm can achieve better than $(1 - \frac{1}{e})$, assuming $P \neq NP$. For maximizing a monotone submodular function under a general matroid constraint, the simple greedy algorithm is $\frac{1}{2}$ -approximate [52]. Călinescu et al. [12] found an algorithm that is $(1 - \frac{1}{e})$ -approximate. Recently, Filmus and Ward [29] introduced a simpler algorithm with the same ratio. Finally, for non-monotone objectives under a matroid constraint, the best known approximation ratio is $\frac{1}{e}$ [26], and the best hardness result is 0.478 [34]. For maximizing a non-monotone submodular function with multiple matroid constraints, Lee et al. [47] presented a $\frac{1}{t+2+\frac{1}{t}+\varepsilon}$ -

approximate algorithm under l matroid constraints.

Our work is also closely related to the literature of submodular matroid secretary problem, which can be formulated as online submodular maximization without free disposal but assuming the elements arrive in random order. The submodular secretary problem has been widely studied recently [6, 26, 39, 48], and constant-competitive algorithms have been found on some special cases, such as on a uniform matroid constraint [7], or when the objective function is to maximize the largest weighted element in the set [32, 42]. However, there is no constant ratio for the general submodular matroid secretary problem till now. Feldman and Zenklusen [28] reduced the problem to the matroid secretary problem with linear objective functions, which implies an $O(\log \log(\text{rank}))$ -competitive algorithm for the submodular matroid secretary problem, matching the current best result for the matroid secretary problem [46, 27].

There is a long line of research on the online bipartite cardinality matching problem [44, 35, 8, 43, 49], and the vertex-weighted version [2, 19]. $(1 - \frac{1}{e})$ -competitive algorithms are known for both cases. For the most general edge-weighted version, random arrival order or free-disposal is necessary for any non-trivial competitive ratio. When online nodes arrive in random order, Wang et al. [55] discovered an algorithm that is $(1 - \frac{1}{e})$ -competitive; while in the free-disposal model, the same competitive ratio can only be achieved by assuming that the offline nodes have large capacity [25, 18], or under the small bid assumption [55]. It remains an important open question whether there is an online algorithm with a competitive ratio unconditionally strictly better than $\frac{1}{2}$ for the free-disposal model of the edge-weighted problem.

Finally, the buyback problem is similar to our model, in which there is a cost associated with revocations. The most related work in this literature is by Babaioff et al. [4, 5], in which a matroid constraint is considered. For other buyback works, see e.g. [17, 40, 41, 53, 3].

2 Preliminaries

We consider elements coming from some ground set Ω , and a non-negative submodular function $f : 2^\Omega \rightarrow \mathbb{R}_+$. We further assume that f is monotone, i.e., $S \subseteq T$ implies that $f(S) \leq f(T)$. Moreover, we assume that there is a matroid (Ω, \mathcal{I}) , and, without loss of generality, every singleton in Ω is independent. Given $a \in \Omega$ and $S \subseteq \Omega$, we denote $S - a := S \setminus \{a\}$, $S + a := S \cup \{a\}$ and $f(a|S) := f(S + a) - f(S)$. We assume value oracle access to the function f and independence oracle access to the matroid, i.e., given a subset $S \subseteq \Omega$, an oracle returns the value $f(S)$ and answers whether $S \in \mathcal{I}$. For

a positive integer n , we denote $[n] := \{1, 2, \dots, n\}$.

Online maximization problem with free disposal.

The algorithm maintains an independent set S , which is initially empty. Elements from Ω arrive in a finite sequence, whose length is not known by the algorithm. In each round when an element u arrives, the algorithm may remove some elements from S , and may also include the current element u into S , as long as S remains independent in \mathcal{I} . The objective is to maximize $f(S)$ at the end of the sequence. We denote by OPT an independent subset of elements in the sequence that maximizes the function f . An algorithm has a competitive ratio $r \leq 1$, if at the end of every finite sequence, the set S satisfies $\mathbf{E}[f(S)] \geq r \cdot f(\text{OPT})$.

Our deterministic algorithm in Section 3 is strictly monotone in the sense that it accepts an arriving item only if there is absolute advantage in doing so. This is formalized in Definition 2.1. Our hardness results in Section 5 and 6 apply to any strictly monotone algorithms.

DEFINITION 2.1. (STRICT MONOTONICITY) *An algorithm is strictly monotone if, in each round, the algorithm includes the new arriving element into the feasible set S (and possibly removing some elements from S) only if the objective value $f(S)$ strictly increases compared to its value at the beginning of the round.*

Auxiliary Set and Weight Function.

Suppose we consider some algorithm. Recall that the algorithm maintains some independent set S . To facilitate the analysis, at the end of each round, we consider an auxiliary set A that keeps track of all the elements that have ever been added into S , but might have been removed since then. For an element u in the sequence, at the beginning of the round in which u arrives, let $S(u)$ be the independent set maintained by the algorithm at this moment, and let $A(u)$ be the set of elements that have been added into S (but might have been removed since then) at this moment.

We remark that the sets $S(u)$ and $A(u)$ are dependent on the algorithm, and so are the following quantities. We denote $w(u) := f(u|A(u))$ as the marginal value of u when it arrives with respect to all the elements that have ever been picked by the algorithm at this moment. For some element u that was added in some previous round, we measure its value with respect to the current set S by $w_S(u) := f(u|A(u) \cap S)$. In general, given a weight function $\omega : \Omega \rightarrow \mathbb{R}$, for $T \subseteq \Omega$, we denote $\omega(T) := \sum_{u \in T} \omega(u)$.

Element Naming Convention. For $i \geq 1$, let u_i denote the i -th element added to S by the algorithm (not the element arriving in the i -th round).

Let S_i and A_i denote the sets of elements contained

in S and A respectively, where S and A are those at the end of the round in which u_i arrives. We denote the value function $w_i := w_{S_i}$.

LEMMA 2.1. (RELATING w AND f) *The functions f and w satisfy the following for each $n \geq 1$.*

- (a) $w(S_n) \leq w_n(S_n) = f(S_n) - f(\emptyset)$.
- (b) $w(A_n) = f(A_n) - f(\emptyset)$.

Proof. For statement (a), the inequality follows because for each $v \in S_n$, $w(v) = f(v|A(v)) \leq f(v|A(v) \cap S_n) = w_n(v)$, where the inequality holds because f is submodular. If we write the elements of $S_n = \{v_1, v_2, v_3, \dots\}$ in the order they arrive, then for each $v_i \in S_n$, we have $w_n(v_i) = f(v_i|A(v_i) \cap S_n) = f(v_i|\{v_1, \dots, v_{i-1}\})$. Hence, a telescoping sum gives $w_n(S_n) = f(S_n) - f(\emptyset)$.

For statement (b), we write the elements of $A_n = \{u_1, u_2, \dots, u_n\}$ in the order they arrive, and observe that $w(u_i) = f(u_i|A(u_i)) = f(u_i|\{u_1, \dots, u_{i-1}\})$. Hence, a similar telescoping sum gives $w(A_n) = f(A_n) - f(\emptyset)$. ■

LEMMA 2.2. (MONOTONE $w_S(u)$) *Suppose an element u arrives in some round. Then, in subsequent rounds, the value $w_S(u)$ does not decrease, when S is modified by the algorithm.*

Proof. Observe that in each round, the algorithm may remove elements from S , and may add the new element to S . Since the elements in the sequence are distinct, it follows that in the subsequent rounds after u arrives, the set $A(u) \cap S$ can only shrink. Since f is submodular, it follows that $w_S(u) := f(u|A(u) \cap S)$ does not decrease, as the algorithm updates S . ■

3 Improved Algorithm for k -Uniform Matroid

In this section, we consider the special case of a k -uniform matroid, i.e., a set is independent iff its cardinality is at most k . Observe that the trivial algorithm that keeps the singleton with the largest value achieves ratio $\frac{1}{k}$. Since we wish to obtain a ratio better than $\frac{1}{4}$, we consider $k \geq 4$ in this section.

Defining α_k . We define α_k to be the unique root in the interval $(3, 4)$ of the equation $a = (1 + \frac{a-2}{k+1})^{k+1}$. We shall show that the competitive ratio is $\frac{1}{\alpha_k}$. It can be shown that α_k is decreasing (see Lemma 3.5). Moreover, as k tends to infinity, the equation defining α_k becomes $a = e^{a-2}$, which has root $\alpha_\infty \approx 3.14619$. For simplicity, we write $\alpha := \alpha_k$ in the rest of this section.

Replacement Condition. The replacement condition is $w(u) > \frac{1}{k}(\alpha \cdot w_S(S) - w(A))$. This means that even when $|S| < k$, if the arriving element u does not have enough value $w(u)$, then it will not be accepted.

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1 Initialize  $S$  and  $A$  to empty sets.
2 for each round when  $u$  arrives do
3   if  $w(u) > \frac{1}{k}(\alpha \cdot w_S(S) - w(A))$  then
4     if  $|S| = k$  then
5        $u' := \arg \min_{v \in S} w_S(v)$ ; (element in  $S$ 
        to be replaced)
6     else
7        $u' := \perp$ ; (no element in  $S$  is replaced)
8      $S \leftarrow S - u' + u$ ;  $A \leftarrow A + u$ ;

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Algorithm 1: Modified Algorithm for k -Uniform Matroids

When the algorithm decides to accept u , if $|S| = k$, then the element u' in S with minimum value under w_S will be replaced; if $|S| < k$, for notational convenience, we set u' to a dummy element \perp . The function f can be extended naturally such that any dummy elements are ignored, and so $w_S(\perp) = 0$.

An important technical result is the following Lemma 3.1, which we use to argue about certain monotone properties of our algorithm. Intuitively, it says that we only accept an element if it is significantly better than the replaced one. We defer its proof in Section 3.1.

LEMMA 3.1. (MONOTONE REPLACEMENT) *Suppose at the beginning of the round when u arrives, the feasible set is S , and Algorithm 1 includes u and discards u' (which could be \perp) from S . Then, $w(u) > \frac{\alpha}{\alpha-1} \cdot w_S(u')$. Observe this implies that $w(u) > w_S(u')$.*

Using Lemma 3.1, by showing the following Lemma 3.2, we can conclude that Algorithm 1 is strictly monotone.

LEMMA 3.2. (MONOTONICITY) *Suppose in each round the algorithm only replaces u' (which could be \perp) in S with the new u such that $w(u) > w_S(u')$. Then, for any $n \geq 0$, $f(S_{n+1}) > f(S_n)$.*

Proof. We write $u = u_{n+1}$ and denote $S_{n+1} = S_n - u' + u$, for some $u' \in S_n$. (We use the convention that a dummy element $\perp \in S_n$ and $A(\perp) = \emptyset$.)

Then, we have $f(S_{n+1}) = f(S_n - u') + f(u|S_n - u') \geq f(S_n - u') + f(u|A(u))$, where the inequality follows because $S_n - u' \subseteq A(u)$ and f is submodular.

We next observe that $w(u) = f(u|A(u))$. Moreover, by the hypothesis of the lemma, $w(u) > w_n(u') = f(u'|A(u') \cap S_n)$, because the algorithm replaces u' by u . Moreover, since $A(u') \cap S_n \subseteq S_n - u'$ and f is submodular, we have $f(u'|A(u') \cap S_n) \geq f(u'|S_n - u')$.

Combining all the inequalities, we have $f(S_{n+1}) \geq f(S_n - u') + f(u|A(u)) > f(S_n - u') + f(u'|A(u') \cap S_n) \geq f(S_n - u') + f(u'|S_n - u') = f(S_n)$, as required. ■

Using Lemma 3.1, we also show the following monotone property, which is useful in proving the competitive ratio.

LEMMA 3.3. (MONOTONE THRESHOLD) *The sequence $\{\alpha \cdot w_n(S_n) - w(A_n)\}_{n \geq 0}$ is monotonically increasing.*

Proof. Fix $n \geq 0$. Observe that $A_{n+1} = A_n + u_{n+1}$, and $w(A_{n+1}) - w(A_n) = w(u_{n+1})$. Hence, to prove $\alpha \cdot w_{n+1}(S_{n+1}) - w(A_{n+1}) \geq \alpha \cdot w_n(S_n) - w(A_n)$, it suffices to show that $\alpha \cdot (w_{n+1}(S_{n+1}) - w_n(S_n)) \geq w(u_{n+1})$.

We write $S_{n+1} = S_n - u' + u_{n+1}$, where $u' \in S_n$. (Again, by convention, if $u' = \perp$ is dummy, we assume $u' \in S_n$.)

Observe that $\alpha \cdot (w_{n+1}(S_{n+1}) - w_n(S_n)) \geq \alpha \cdot (w_{n+1}(u_{n+1}) - w_n(u')) \geq \alpha \cdot (w(u_{n+1}) - w_n(u'))$, where the first inequality follows from $w_{n+1} \geq w_n$ and the second follows from $w_{n+1}(u_{n+1}) \geq w(u_{n+1})$ (both of which follows from the submodularity of f). Finally, Lemma 3.1 implies that $\alpha \cdot (w(u_{n+1}) - w_n(u')) \geq w(u_{n+1})$, which completes the proof. ■

THEOREM 3.1. *Algorithm 1 is $\frac{1}{\alpha}$ -competitive.*

Proof. We suppose that the algorithm has included n elements into A . Then, the feasible solution at the end is S_n , and we have $f(\text{OPT}) \leq f(A_n \cup \text{OPT}) \leq f(A_n) + \sum_{u \in \widehat{\text{OPT}}} f(u|A_n)$, where $\widehat{\text{OPT}} := \text{OPT} \setminus A_n$ are the elements in an optimal solution that are discarded immediately in the rounds that they arrive.

For $u \in \widehat{\text{OPT}}$, by the submodularity of f , $f(u|A_n) \leq w(u)$, which, since u is discarded in the round it arrives, is at most $\frac{1}{k}(\alpha \cdot w_{S(u)}(S(u)) - w(A(u)))$. This quantity is at most $\frac{1}{k}(\alpha \cdot w_n(S_n) - w(A_n))$, by Lemma 3.3.

Since $|\widehat{\text{OPT}}| \leq k$, we have $f(\text{OPT}) \leq f(A_n) + \alpha \cdot w_n(S_n) - w(A_n) \leq \alpha \cdot f(S_n) - (\alpha - 1) \cdot f(\emptyset) \leq \alpha \cdot f(S_n)$, where the second inequality follows from Lemma 2.1. ■

COROLLARY 3.1. *For monotone f with uniform matroid, there exists a deterministic algorithm with competitive ratio at least $\min_k \max\{\frac{1}{k}, \frac{1}{\alpha_k}\} = \frac{1}{\alpha_4} > 0.2959$.*

3.1 Proof of Lemma 3.1 Define $\beta := \alpha^{\frac{1}{k+1}} = 1 + \frac{\alpha-2}{k+1}$. One can check that $k\beta^{k+1} - (\alpha+k-1)\beta^k + \alpha = 0$.

For ease of notation, we assume that there are k dummy elements $\{u_1, u_2, \dots, u_k\}$. The function f is extended naturally such that any dummy elements are ignored. For $1 \leq i \leq k$, we use the convention that $S_i = A_i = \{u_1, u_2, \dots, u_i\}$. Therefore, the real algorithm starts at $n = k + 1$.

We prove a stronger statement that for all $n \geq k$, we have the following.

(A) If u_{n+1} replaces $u' \in S_n$ (which could be dummy), then $w(u_{n+1}) > \frac{\alpha}{\alpha-1} \cdot w_n(u')$.

(B) $w(A_{n+1}) \geq \beta \cdot w(A_n)$.

Observe that the first k dummy elements ensure that $w(A_i) = 0$ for $0 \leq i \leq k$, and hence statement (B) actually holds for $0 \leq n < k$.

For contradiction's sake, we consider the smallest integer n (at least k) for which at least one of the above statements does not hold.

We next prove the following claim.

Claim. For all $I \subseteq A_n$ such that $|I| \leq k$, $w_n(S_n) \geq w(I)$.

Proof. By our assumption, for all $k \leq i < n$, if u_{i+1} replaces u'_{i+1} , then $w(u_{i+1}) > \frac{\alpha}{\alpha-1} \cdot w_i(u'_{i+1})$, where u'_{i+1} is an element attaining $\min_{u \in S_i} w_i(u)$.

Observe that if an element u stays in the set S_j (for $j \geq i + 1$), then $w_j(u)$ does not decrease as j increases (Lemma 2.2). Moreover, observe that $\min_{v \in S_j} w_j(v)$ is non-decreasing as j increases. Hence, for any element $u \in S_j$, we must have $w_j(u) \geq w_i(u'_{i+1}) \geq w(u'_{i+1})$.

Hence, it follows that if $u \in S_n$ and u' is an element that is replaced at some point, then $w_n(u) \geq w(u')$.

Hence, if we set $P := S_n \cap I$, we have $w_n(S_n) = w_n(P) + w_n(S_n \setminus P) \geq w(P) + w(I \setminus P) = w(I)$, as required. ■

Hence, we can pick $I = A_n \setminus A_{n-k}$, and have $w_n(S_n) \geq w(A_n) - w(A_{n-k}) \geq (1 - \frac{1}{\beta^k})w(A_n)$, where the last inequality holds because $w(A_{i+1}) \geq \beta \cdot w(A_i)$ holds for $i < n$. Since the algorithm replaces an element from S_n with u_{n+1} , we have $w(u_{n+1}) > \frac{1}{k}(\alpha \cdot w_n(S_n) - w(A_n))$. Combining this with the above lower bound for $w_n(S_n)$, we have:

$$(3.1) \quad \begin{aligned} w(u_{n+1}) &> \frac{1}{k} \cdot \{\alpha(1 - \frac{1}{\beta^k}) - 1\} \cdot w(A_n) \\ &= (\beta - 1) \cdot w(A_n), \end{aligned}$$

where the last equality follows from the choice of β .

We first show that statement (B) must hold for n . From $w(A_{n+1}) - w(A_n) = w(u_{n+1})$ and inequality (3.1), we have $w(A_{n+1}) \geq \beta \cdot w(A_n)$, as required. It remains to show that statement (A) must also hold for n . We prove the following lemma.

LEMMA 3.4. *For all $0 \leq i \leq k - 1$, $\min_{u \in S_n} w_n(u) \leq \frac{w_{n-i}(S_{n-i})}{k-i}$.*

Proof. The claim holds trivially for $i = 0$. We next fix $i > 0$.

We next show that for $j < n$, $w_{j+1}(S_{j+1}) - w_j(S_j) \leq w_{j+1}(u_{j+1})$. When $S_{j+1} = S_j + u_{j+1}$, we have $w_j(S_j) = w_{j+1}(S_j)$, and so equality holds. Suppose

$S_{j+1} = S_j - u' + u_{n+1}$ for some $u' \in S_j$. Then, from Lemma 2.1, $w_{j+1}(S_{j+1}) - w_j(S_j) = f(S_{j+1}) - f(S_j) = f(S_j - u' + u_{j+1}) - f(S_j) \leq f(S_j + u_{j+1}) - f(S_j) = f(u_{j+1}|S_j) \leq f(u_{j+1}|S_j - u') = w_{j+1}(u_{j+1})$, where the first inequality follows from the monotonicity of f .

Hence, summing the above inequality over appropriate indices, we have

$$\sum_{j=0}^{i-1} (w_{n-j}(S_{n-j}) - w_{n-j-1}(S_{n-j-1})) \leq \sum_{j=0}^{i-1} w_{n-j}(u_{n-j}).$$

After rearranging, we have $w_{n-i}(S_{n-i}) \geq w_n(S_n) - \sum_{j=0}^{i-1} w_{n-j}(u_{n-j})$.

Define $P = \{0 \leq j \leq i-1 : u_{n-j} \in S_n\}$ and $Q = \{0, 1, \dots, i-1\} \setminus P$.

By Lemma 2.2, for $j \in P$, $w_n(u_{n-j}) \geq w_{n-j}(u_{n-j})$. Hence, we have $w_n(S_n) - \sum_{j=0}^{i-1} w_{n-j}(u_{n-j}) \geq w_n(S_n - \{u_{n-j}\}_{j \in P}) - \sum_{j \in Q} w_{n-j}(u_{n-j})$.

Fix $j \in Q$. Observe that in some round l , u_{n-j} is replaced by some element u , where $w_{n-j}(u_{n-j}) \leq w_{l-1}(u_{n-j}) \leq \min_{v \in S_l} w_l(v)$. The first inequality comes from Lemma 2.2. Moreover, the minimum weight is only increasing during the execution of our algorithm, because when the algorithm needs to replace an element in S , it will choose $\arg \min_{v \in S} w_S(v)$. Hence, it follows that $w_{n-j}(u_{n-j}) \leq \min_{v \in S_n} w_n(v)$. Therefore,

$$\begin{aligned} w_{n-i}(S_{n-i}) &\geq w_n(S_n - \{u_{n-j}\}_{j \in P}) - \sum_{j \in Q} w_{n-j}(u_{n-j}) \\ &\geq (|S_n| - |P| - |Q|) \cdot \min_{u \in S_n} w_n(u) \\ &= (k - i) \cdot \min_{v \in S_n} w_n(v), \end{aligned}$$

as required. \blacksquare

Proving Statement (A). Define $\gamma := \frac{(\alpha-2)(\alpha-1)}{\alpha} \cdot \frac{k}{k+1}$. Observe that $\alpha\gamma > 2$ (see Lemma 3.6(a)).

The easy case is when $w_n(S_n) \leq \gamma \cdot w(A_n)$. Then, from (3.1), we have $w(u_{n+1}) > (\beta - 1) \cdot w(A_n) \geq \frac{\beta-1}{\gamma} \cdot w(S_n) = \frac{\alpha}{\alpha-1} \cdot \frac{w(S_n)}{k} \geq \frac{\alpha}{\alpha-1} \cdot \min_{v \in S_n} w_n(v)$, where the last inequality comes from Lemma 3.4. Hence, we can assume $w_n(S_n) > \gamma \cdot w(A_n)$ from now on. Recall that since u_{n+1} is selected by the algorithm, we have $w(u_{n+1}) > \frac{1}{k} \cdot (\alpha w_n(S_n) - w(A_n)) \geq \frac{1}{k} \cdot (\alpha\gamma - 1) \cdot w(A_n)$. Hence, we next give a lower bound on $w(A_n)$ with respect to $m := \min_{u \in S_n} w_n(u)$.

Suppose $0 < i \leq n$ is the smallest integer such that $w_{n-i}(S_{n-i}) \leq \gamma \cdot w(A_{n-i})$. Such an integer must exist because the first dummy element implies that $w_1(S_1) = w(A_1) = 0$. For $0 \leq j < i$, we have $w_{n-j}(S_{n-j}) > \gamma \cdot w(A_{n-j})$. Since the algorithm replaces an element from S_{n-j} with u_{n-j+1} , it follows that $w(A_{n-j+1}) - w(A_{n-j}) = w(u_{n-j+1}) \geq \frac{1}{k}(\alpha \cdot w_{n-j}(S_{n-j}) - w(A_{n-j})) \geq \frac{1}{k}(\alpha\gamma - 1) \cdot w(A_{n-j})$.

Define $\delta := 1 + \frac{1}{k}(\alpha\gamma - 1)$. Hence, for $0 \leq j < i$, $w(A_{n-j+1}) \geq \delta \cdot w(A_{n-j})$.

Define the function $\vartheta(x) := (1-x)\delta^{kx}$ for $x \in [0, 1]$, and $\lambda := 1 - \frac{1}{k \ln \delta}$. Observe that ϑ is increasing on $(0, \lambda)$ and decreasing on $(\lambda, 1)$. Hence, ϑ attains its maximum at λ . We consider two cases.

Case 1. $i \leq \lambda k$. In this case, we have

$$\begin{aligned} w(A_n) &\geq \delta^{i-1} \beta \cdot w(A_{n-i}) \geq \frac{\delta^{i-1} \beta}{\gamma} \cdot w_{n-i}(S_{n-i}) \geq \\ &\frac{\beta k}{\delta \gamma} \cdot \delta^i (1 - \frac{i}{k}) \cdot m = \frac{\beta k}{\delta \gamma} \cdot \vartheta(\frac{i}{k}) \cdot m, \end{aligned}$$

where the last inequality follows from Lemma 3.4.

To finish with this case, we have

$$\begin{aligned} w(u_{n+1}) &> \frac{1}{k} \cdot (\alpha\gamma - 1) \cdot w(A_n) \geq \frac{1}{k} \cdot (\alpha\gamma - 1) \cdot \frac{\beta k}{\delta \gamma} \cdot \\ &\vartheta(\frac{i}{k}) \cdot m \geq \frac{\beta}{\delta \gamma} \cdot (\alpha\gamma - 1) \cdot \vartheta(0) \cdot m \geq \frac{\alpha}{\alpha-1} \cdot m, \end{aligned}$$

where the last inequality follows from Lemma 3.6(b).

Case 2. $i > \lambda k$. In this case, set $\ell := \lfloor \lambda k \rfloor$. Then, we have

$$\begin{aligned} w(A_n) &\geq \delta^\ell \cdot w(A_{n-\ell}) \geq \delta^\ell \cdot w_{n-\ell}(S_{n-\ell}) \geq k \cdot \\ &\vartheta(\frac{1}{k} \cdot \lfloor \lambda k \rfloor) \cdot m, \end{aligned}$$

where the last inequality follows from Lemma 3.4, and the penultimate inequality follows from Lemma 2.1 and the monotonicity of f . Hence, to finish with this case, we have

$$w(u_{n+1}) > (\alpha\gamma - 1) \cdot \vartheta(\frac{1}{k} \cdot \lfloor \lambda k \rfloor) \cdot m \geq \frac{\alpha}{\alpha-1} \cdot m,$$

where the last inequality follows from Lemma 3.6(c).

This finishes the proof of statement (A).

LEMMA 3.5. (α_k IS DECREASING) α_k is decreasing with respect to k .

Proof. Recall that α_k is defined to be the unique root of equation $\alpha = (1 + \frac{\alpha-2}{k+1})^{k+1}$ that is in (3, 4).

We are going to prove that for any $1 \leq k \leq k'$, $\alpha_k \geq \alpha_{k'}$. Since $\alpha_k = (1 + \frac{\alpha_k-2}{k+1})^{k+1} \leq (1 + \frac{\alpha_{k'}-2}{k'+1})^{k'+1}$, we have $(1 + \frac{\alpha_k-2}{k+1})^{k+1} - \alpha_k \geq 0 = (1 + \frac{\alpha_{k'}-2}{k'+1})^{k'+1} - \alpha_{k'}$. Observe that the function $(1 + \frac{x-2}{k'+1})^{k'+1} - x$ of x is non-decreasing when $x \geq 3$. Therefore, $\alpha_k \geq \alpha_{k'}$. \blacksquare

LEMMA 3.6. (TECHNICAL INEQUALITIES) We have the following technical inequalities.

- (a) $\alpha\gamma > 2$.
- (b) $\frac{\beta}{\delta \gamma} \cdot (\alpha\gamma - 1) \geq \frac{\alpha}{\alpha-1}$.
- (c) $(\alpha\gamma - 1) \cdot \vartheta(\frac{1}{k} \cdot \lfloor \lambda k \rfloor) \geq \frac{\alpha}{\alpha-1}$.

The proof to Lemma 3.6 is omitted and can be found in the full version[14].

4 Randomized Algorithm for Partition Matroid

We consider $\Omega := \cup_{l \in [L]} \Omega_l$, which is a disjoint union of L sets. Suppose for $l \in [L]$, capacity k_l is associated with the set Ω_l . Then, the *partition matroid* (Ω, \mathcal{I}) is defined such that a set $S \subset \Omega$ is independent in \mathcal{I} iff for all $l \in [L]$, $|S \cap \Omega_l| \leq k_l$. For an element $u \in \Omega$, we

denote $l(u) \in [L]$ such that $u \in \Omega_{l(u)}$. We consider a monotone submodular $f : 2^\Omega \rightarrow \mathbb{R}_+$ objective function.

In this section, we consider randomized algorithms for the online problem. We first define a continuous variant of the problem and describe a corresponding online algorithm. We observe in Section 3 that the competitive ratio for k -uniform matroid is $\frac{1}{\alpha_k}$, where α_k approaches the root $\alpha_\infty \approx 3.14619$ of $a = e^{a-2}$, as k tends to infinity. By considering the continuous variant of the problem, we are essentially considering arbitrarily large k in order to achieve ratio $\frac{1}{\alpha_\infty}$. For simplicity, in this section, we write $\alpha := \alpha_\infty$. Moreover, we shall describe a rounding procedure that gives us an online randomized algorithm for the original problem.

Continuous Variant. The algorithm maintains a vector $S \in \mathbb{R}_+^\Omega$ such that initially $S = \vec{0}$. A vector S is *feasible* (with respect to \mathcal{I}) if for all $l \in [L]$, $\sum_{u \in \Omega_l} S_u \leq k_l$. The interpretation is that we can take a fractional number of copies (even larger than 1) of an item. When an item u arrives, the algorithm may increase the coordinate S_u corresponding to the item u and possibly decrease the coordinates S_v for other items v to maintain feasibility.

The objective function $\hat{f} : \mathbb{R}_+^\Omega \rightarrow \mathbb{R}_+$ is induced by the original function as follows. Given $S \in \mathbb{R}_+^\Omega$, denote $R(S) \subset \Omega$ as the random subset sampled by including each element $u \in \Omega$ independently with probability $1 - \exp(-S_u)$. Then, $\hat{f}(S) := \mathbf{E}[f(R(S))]$.

Measure Interpretation. We also interpret S as a subset of the product measure space $\Omega \times \mathbb{R}_+$ (where Ω has the cardinality measure and \mathbb{R}_+ has the standard Lebesgue measure). Specifically, we identify a vector $S \in \mathbb{R}_+^\Omega$ with the following subset: $\{(u, t) : u \in \Omega \wedge t \in (0, S_u]\}$. (We use half-open intervals to make the rounding description more convenient later.) Observe that there is a natural 1-1 correspondence between vectors in \mathbb{R}_+^Ω and *valid* subsets in $\mathcal{M}(\mathbb{R}_+^\Omega)$ defined as follows.

DEFINITION 4.1. (VALID SUBSET) A subset $B \subset \Omega \times \mathbb{R}_+$ is *valid* if for all $u \in \Omega$, there exists $t_u \geq 0$ such that $\{u\} \times (0, t_u] \subseteq B$ and for all $t > t_u$, $(u, t) \notin B$.

We use $\mathcal{M}(\mathbb{R}_+^\Omega)$ to denote the collection of valid subsets of $\Omega \times \mathbb{R}_+$.

Observe that valid subsets in $\mathcal{M}(\mathbb{R}_+^\Omega)$ are closed under union and intersection. Hence, it makes sense to consider the submodularity of the function \hat{f} interpreted as having domain $\mathcal{M}(\mathbb{R}_+^\Omega)$.

LEMMA 4.1. (MONOTONICITY AND SUBMODULARITY OF \hat{f}) Suppose $f : 2^\Omega \rightarrow \mathbb{R}_+$ is monotone and submodular. Then, $\hat{f} : \mathcal{M}(\mathbb{R}_+^\Omega) \rightarrow \mathbb{R}_+$ is also monotone and submodular.

Proof. This can be proved by a coupling argument. Suppose ω is sampled from $[0, 1]^\Omega$ uniformly at random. Given $P \in \mathcal{M}(\mathbb{R}_+^\Omega)$, denote $R_\omega(P) := \{u \in \Omega : \omega_u \leq 1 - \exp(-P_u)\}$. Hence, it follows that $\hat{f}(P) = \mathbf{E}_\omega[f(R_\omega(P))]$.

Then, the results follows because of the following facts that can be verified easily for any $P, Q \in \mathcal{M}(\mathbb{R}_+^\Omega)$ and $\omega \in \Omega$.

1. If $P \subseteq Q$, then $R_\omega(P) \subseteq R_\omega(Q)$.
2. $R_\omega(P \cup Q) = R_\omega(P) \cup R_\omega(Q)$.
3. $R_\omega(P \cap Q) = R_\omega(P) \cap R_\omega(Q)$.

Hence, the monotonicity and submodularity of \hat{f} follow from those of f immediately. \blacksquare

It will be clear from the context whether we use the vector or the measure interpretation for S . For instance, S_u is the u -th coordinate of the vector, and $(u, t) \in S$ means that $S_u \geq t$. We use $\|S\|$ to denote the measure of $S \in \mathcal{M}(\mathbb{R}_+^\Omega)$. For $l \in [L]$, we denote $S|_{\Omega_l} := \{(u, t) \in S : u \in \Omega_l\}$. Then, the feasibility of S can be expressed as $\|S|_{\Omega_l}\| \leq k_l$ for all $l \in [L]$.

Increment. Given valid $B \in \mathcal{M}(\mathbb{R}_+^\Omega)$, an element $u \in \Omega$ and $t \geq 0$, we use $B \oplus (u, [t]) := B \cup (\{u\} \times (B(u), B(u) + t])$ to denote adding extra t units of element u to B .

Marginal Value. Given valid $B', B \in \mathcal{M}(\mathbb{R}_+^\Omega)$, we denote the marginal value $\hat{f}(B'|B) := \hat{f}(B' \cup B) - \hat{f}(B)$. The marginal value of an element $u \in \Omega$ with respect to B is $\hat{f}(u|B) := \lim_{t \rightarrow 0^+} \frac{\hat{f}(B \oplus (u, [t])) - \hat{f}(B)}{t}$.

Auxiliary Set A . Observe that as the algorithm increases S_u from 0 to some value t , we can interpret this as adding (u, τ) to S continuously for τ from 0 to t . Similarly, as the algorithm decreases S_v from t_2 to t_1 , we can interpret this as removing (v, τ) from S continuously for τ from t_2 to t_1 . While the algorithm modifies S , we use an auxiliary set A to keep track of all pairs (u, t) that have ever been added to S , but could have already been removed at some point.

Value Function w . Suppose in the round that u arrives, the algorithm has so far increased S_u to some value $t \geq 0$. In order to decide whether to further increase S_u , we denote $z = (u, t)$ and use a value function $w(z) := \hat{f}(u|A(z))$, where $A(z)$ is the set of pairs that have ever been added to S by the algorithm up to this moment. Observe that w is dependent on the behavior of the algorithm, and can be interpreted as a function $w : A \rightarrow \mathbb{R}_+$, where A is the auxiliary set. Hence, for any subset $B \subseteq A$, we denote $w(B) := \int_B w(z) dz$ as the Lebesgue integral.

LEMMA 4.2. (RELATING w AND \hat{f}) Suppose at some instant, S is the feasible set maintained by the (continuous) algorithm, and A is the auxiliary set defined above

in the same instant. Then, the following holds.

- (a) $w(S) \leq \widehat{f}(S) - \widehat{f}(\emptyset)$.
- (b) $w(A) = \widehat{f}(A) - \widehat{f}(\emptyset)$.

Proof. We treat the measure $\|A\|$ as a way to keep track of time τ .

For statement (a), for $z = (u, t) \in S$, $w(z) = \widehat{f}(u|A(z)) \leq \widehat{f}(u|A(z) \cap S)$, where the last inequality follows from the submodularity of \widehat{f} . Hence, integrating over $z \in S$, we have $w(S) \leq \widehat{f}(S) - \widehat{f}(\emptyset)$.

For statement (b), for $z = (u, t) \in A$, $w(z) = \widehat{f}(u|A(z))$. Hence, integrating over $z \in A$ gives $w(A) = \widehat{f}(A) - \widehat{f}(\emptyset)$. ■

DEFINITION 4.2. (KNAPSACK FOR ROUNDING) To facilitate the description of the rounding procedure, we can view the algorithm as storing the pairs in S in a knapsack $\mathbf{B} := \cup_{l \in [L]} \{l\} \times (0, k_l]$, where each interval $(0, k_l]$ is also equipped with the Lebesgue measure. Specifically, when pairs (u, t) are added to S continuously (and other pairs possibly removed), we associate (u, t) with a point $\varphi(u, t) \in \mathbf{B}$ such that the following conditions hold.

1. Element $u \in \Omega_{l(u)}$ is put in the correct part, i.e., $\varphi(u, t) = (l(u), s)$, for some $s \in (0, k_{l(u)}]$.
2. At any moment, $\varphi|_S : S \rightarrow \mathbf{B}$ is injective. (Half-intervals are used to satisfy this property.)
3. For any (measurable) subset $B \subseteq S$, $\|\varphi(B)\| = \|B\|$.

We remark that there is a natural way to replace pairs in S and assign values to φ such that the above conditions hold. Hence, in the description of the algorithm, we do not explicitly mention φ .

DEFINITION 4.3. (ONLINE ROUNDING PROCEDURE) Before any item arrives, a random subset $\mathcal{Z} \in \mathbf{B}$ is sampled in the knapsack as follows. For each $l \in [L]$, k_l points are sampled uniformly at random independently from $\{l\} \times (0, k_l]$ and included in \mathcal{Z} .

At any moment when the (continuous) algorithm is maintaining $S \in \mathcal{M}(\mathbb{R}_+^\Omega)$, we can imagine that the randomized algorithm (which must maintain feasibility in the original partition matroid (Ω, \mathcal{I})) is keeping $\tilde{S} := \{u \in \Omega : \exists t, (u, t) \in S \wedge \varphi(u, t) \in \mathcal{Z}\}$.

THEOREM 4.1. (RANDOMIZED ONLINE ALGORITHM FOR PARTITION MATROID) The rounding procedure in Definition 4.3 can be applied to Algorithm 2 to produce a randomized algorithm with competitive ratio $\frac{1}{\alpha_\infty}$.

4.1 Continuous Online Algorithm for Partition Matroid Algorithm Model. Without loss of generality, we assume that the algorithm knows the capacity k_l for each part Ω_l , and when an element u arrives,

it also knows to which part $l(u)$ the element belongs. This is because the algorithm can keep on accepting elements until a conflict is detected, at which point it can tell which elements are in the full part and its capacity. We assume that oracle accesses to the objective function $\widehat{f}(\cdot)$ and its marginals $\widehat{f}(\cdot|\cdot)$ (which involves first derivatives). For ease of exposition, we do not discuss how these quantities can be approximated by sampling the original function f . Moreover, we assume that the algorithm can monitor and change variables continuously.

LEMMA 4.3. (CONTINUOUS REPLACEMENT) Suppose during the round that $u \in \Omega_l$ arrives, S_u is currently being increased, i.e., $w(u, S_u) > \frac{1}{k_l} (\alpha \cdot w(S|_{\Omega_l}) - w(A|_{\Omega_l}))$. Moreover, suppose at this moment $\|S|_{\Omega_l}\| = k_l$. Then, we have

$$w(u, S_u) > \frac{\alpha}{\alpha - 1} \min_{z \in S|_{\Omega_l}} w(z).$$

LEMMA 4.4. (MONOTONE THRESHOLD) For each $l \in [L]$, the quantity $(\alpha \cdot w(S|_{\Omega_l}) - w(A|_{\Omega_l}))$ is monotonically increasing during the execution of algorithm.

Proof. Fix $l \in [L]$. We use the parameter $\tau := \|A|_{\Omega_l}\|$ to keep track of time. (Observe that τ does not change if elements from other $\Omega_{l'}$'s are considered.) Define $G(\tau)$ as the quantity $\alpha \cdot w(S|_{\Omega_l}) - w(A|_{\Omega_l})$ at the instant when $\|A|_{\Omega_l}\| = \tau$. Suppose at the instant τ , for some $u \in \Omega_l$, a pair $z = (u, t)$ is being included into S , i.e., S_u is increasing and τ is moving forward. Hence, $w(A|_{\Omega_l})$ is increasing at rate $w(z)$.

If at this moment $\|S|_{\Omega_l}\| < k_l$, then no pair is being removed from S , and we have $G'(\tau) = (\alpha - 1) \cdot w(z) \geq 0$.

Otherwise, pairs with value $\mathbf{m} := \min_{z' \in S|_{\Omega_l}} w(z')$ are being removed from $S|_{\Omega_l}$. Hence, $G'(\tau) = \alpha(w(z) - \mathbf{m}) - w(z) \geq \alpha(w(z) - \frac{\alpha-1}{\alpha} \cdot w(z)) - w(z) = 0$, where the inequality follows from Lemma 4.3. ■

The next lemma compares the objective value of a subset $O \subset \Omega$ with that of a valid set $A \in \mathcal{M}(\mathbb{R}_+^\Omega)$.

LEMMA 4.5. Suppose f is monotone and submodular. Then, for any finite $O \subset \Omega$ and valid $A \in \mathcal{M}(\mathbb{R}_+^\Omega)$, we have

$$f(O) \leq \widehat{f}(A) + \sum_{v \in O} \widehat{f}(v|A).$$

Proof. We prove by induction on the cardinality of O . The statement holds trivially when $|O| = 0$, because f is monotone.

Fix $u \in O$, and let $O' = O - u$. We assume the statement holds for O' . Define $g : 2^\Omega \rightarrow \mathbb{R}_+$ as $g(X) := f(X + u)$, which is also monotone and submodular.

```

1 Initialize  $S$  and  $A$  to empty.
2 for each round when  $u$  in  $\Omega_l$  arrives do
   /* We try to increase  $S_u$  by including pairs  $(u, t)$  in  $S$  and possibly decrease other  $S_v$ 
   by removing some pairs from  $S$ . We use the parameter  $t = S_u$  to keep track of how
   much we increase  $S_u$ . */
3   At the beginning of the round,  $t = 0$ .
4   while  $w(u, S_u) > \frac{1}{k_l} (\alpha \cdot w(S|\Omega_l) - w(A|\Omega_l))$  do
5     Increase  $S_u = t$  and  $A_u = t$  (by including  $(u, t)$ ) continuously.
6     if  $\|S|\Omega_l\| = k_l$  then
       /* We need to remove some pairs from  $S$  to maintain feasibility. */
7       Define  $T := \{v \in \Omega_l : (v, S_v) \in \arg \min_{z \in S|\Omega_l} w(z)\}$ . /*  $T$  can change as  $S$  changes;
       observe that  $v \in T$  implies that  $S_v > 0$ . */
8       For  $v \in T$ , we simultaneously decrease  $S_v$  (i.e., removing  $(v, S_v)$  from  $S$ ) at an appropriate rate
       such that as  $S$  changes, no element leaves  $T$  unless some  $v \in T$  has  $S_v$  dropping to 0; on the
       other hand, it is possible that some new element can join  $T$  as  $S$  changes.
9       More precisely, define  $w_v(\tau) := w(v, \tau)$ , whose value was defined earlier at the moment when
        $(v, \tau)$  was included in  $S$  and  $A$ . Observe that  $w_v(\tau)$  is a decreasing function of  $\tau$ .
10      Let  $\eta_v := |w'_v(S_v)|$ . Then, for each  $v \in T$ ,  $S_v$  is decreased at rate (with respect to  $t$ ) given by
        $\frac{dS_v}{dt} = -\frac{\eta_v^{-1}}{\sum_{s \in T} \eta_s^{-1}}$ .

```

Algorithm 2: Continuous Online Algorithm for Partition Matroids

For $v \in O$, define $A^v := \{(x, t) \in A : x = v\}$ and have
 $A^{-v} := \{(x, t) \in A : x \neq v\}$. Moreover, we have

$$\begin{aligned}
\widehat{f}(A) &= \widehat{f}(A^v \cup A^{-v}) \\
&= (1 - e^{-A_v}) \mathbf{E}[f(\mathbf{R}(A^{-v}) + v)] \\
&\quad + e^{-A_v} \mathbf{E}[f(\mathbf{R}(A^{-v}))] \\
(4.2) \quad &= (1 - e^{-A_v}) \mathbf{E}[f(v|\mathbf{R}(A^{-v}))] + \mathbf{E}[f(\mathbf{R}(A^{-v}))],
\end{aligned}$$

Hence, we can interpret $\widehat{f}(A)$ as a function of A_v :

$$\tau \mapsto (1 - e^{-\tau}) \mathbf{E}[f(v|\mathbf{R}(A^{-v}))] + \mathbf{E}[f(\mathbf{R}(A^{-v}))].$$

Differentiating this function with respect to τ , we have the following claim.

Claim. For any $v \in \Omega$, $\widehat{f}(v|A) = e^{-A_v} \mathbf{E}[f(v|\mathbf{R}(A^{-v}))]$.

In particular, for $v \neq u$, we have

$$\begin{aligned}
\widehat{f}(v|A) &= e^{-A_v} \mathbf{E}[f(v|\mathbf{R}(A^{-v}))] \\
(4.3) \quad &\geq e^{-A_v} \mathbf{E}[f(v|\mathbf{R}(A^{-v}) + u)] = \widehat{g}(v|A^{-u}),
\end{aligned}$$

where the inequality follows from the submodularity of f . Hence, using (4.2) for the first equality below, we

$$\begin{aligned}
\widehat{f}(A) + \sum_{v \in O} \widehat{f}(v|A) &= (1 - e^{-A_u}) \mathbf{E}[f(u|\mathbf{R}(A^{-u}))] \\
&\quad + \mathbf{E}[f(\mathbf{R}(A^{-u}))] + \widehat{f}(u|A) + \sum_{v \in O'} \widehat{f}(v|A) \\
&= \mathbf{E}[f(u|\mathbf{R}(A^{-u}))] + \mathbf{E}[f(\mathbf{R}(A^{-u}))] \\
&\quad + \sum_{v \in O'} \widehat{f}(v|A) \\
&= \mathbf{E}[f(\mathbf{R}(A^{-u}) + u)] + \sum_{v \in O'} \widehat{f}(v|A) \\
&= \widehat{g}(A^{-u}) + \sum_{v \in O'} \widehat{f}(v|A) \\
&\geq \widehat{g}(A^{-u}) + \sum_{v \in O'} \widehat{g}(v|A^{-u}) \\
&\geq g(O') = f(O),
\end{aligned}$$

where the first inequality follows from (4.3) and the last inequality follows from induction hypothesis. ■

LEMMA 4.6. (COMPETITIVE RATIO OF CONTINUOUS ALGORITHM) *Suppose $\text{OPT} \subset \Omega$ is an independent subset of items that have ever arrived, and S is the feasible set maintained by Algorithm 2 at the end. Then, $f(\text{OPT}) \leq \alpha \cdot \widehat{f}(S)$.*

Proof. We use $\tau := \|A\|$ to keep track of time. For instance, we denote $A(\tau)$ as the auxiliary set A at the instant when $\|A\| = \tau$, and denote $S(\tau)$ as the S at the

same instant. We use $\hat{\tau}$ to denote the instant at the end. For $u \in \text{OPT}$, we use τ_u to denote the instant when the algorithm stops including pairs (u, t) involving u .

For $u \in \text{OPT} \cap \Omega_l$, by the submodularity of \hat{f} , $\hat{f}(u|A(\hat{\tau})) \leq \hat{f}(u|A(\tau_u))$, which is at most

$$\frac{1}{k_l}(\alpha \cdot w(S(\tau_u)|_{\Omega_l}) - w(A(\tau_u)|_{\Omega_l})),$$

because the algorithm does not accept pairs involving u after time τ_u . This last quantity is at most $\frac{1}{k_l}(\alpha \cdot w(S(\hat{\tau})|_{\Omega_l}) - w(A(\hat{\tau})|_{\Omega_l}))$, by Lemma 4.4. Using Lemma 4.5, we have

$$f(\text{OPT}) \leq \hat{f}(A(\hat{\tau})) + \sum_{u \in \text{OPT}} \hat{f}(u|A(\hat{\tau})).$$

Since $|\text{OPT} \cap \Omega_l| \leq k_l$, we have

$$\begin{aligned} f(\text{OPT}) &\leq \hat{f}(A(\hat{\tau})) \\ &+ \sum_{l \in [L]} \sum_{u \in \text{OPT} \cap \Omega_l} \frac{1}{k_l}(\alpha \cdot w(S(\hat{\tau})|_{\Omega_l}) - w(A(\hat{\tau})|_{\Omega_l})) \\ &\leq \hat{f}(A(\hat{\tau})) + \sum_{l \in [L]} (\alpha \cdot w(S(\hat{\tau})|_{\Omega_l}) - w(A(\hat{\tau})|_{\Omega_l})) \\ &= \hat{f}(A(\hat{\tau})) + \alpha \cdot w(S(\hat{\tau})) - w(A(\hat{\tau})) \\ &\leq \alpha \cdot \hat{f}(S(\hat{\tau})), \end{aligned}$$

where the last inequality comes from Lemma 4.2. ■

LEMMA 4.7. (ROUNDING PRESERVES RATIO) *Suppose the rounding procedure described in Definition 4.3 takes valid $S \in \mathcal{M}(\mathbb{R}_+^\Omega)$ and produces $\tilde{S} \subset \Omega$. Then, $\mathbf{E}[f(\tilde{S})] \geq \hat{f}(S)$.*

Proof of Theorem 4.1: Lemma 4.6 shows that the competitive ratio of the continuous algorithm is $\frac{1}{\alpha_\infty}$. Lemma 4.7 shows that the rounding procedure can produce a randomized algorithm for the original discrete problem with the same guarantee on the competitive ratio. ■

4.2 Large Replacement: Proof of Lemma 4.3

Suppose we fix $l \in [L]$. For ease of notation, we write $\hat{S} := S|_{\Omega_l}$, $\hat{A} := A|_{\Omega_l}$ and $k := k_l$. We use $\hat{A}(\tau)$ to denote the \hat{A} at the instant when it has measure τ and we use $\hat{S}(\tau)$ to denote the corresponding \hat{S} at the same instant. We can imagine that τ increases as pairs pertaining to Ω_l are added to S and A . To simplify the argument, we imagine that when τ is increased from 0 to k , \hat{A} is filled with dummy pairs such that $\hat{A}(k) = \{\perp\} \times (0, k]$ for some dummy element \perp that has 0 value.

We prove a stronger statement that for $\tau \geq k$, suppose currently there is some $u \in \Omega_l$ such that S_u is being increased. Then, the following holds.

$$(A) \quad w(u, S_u) > \frac{\alpha}{\alpha-1} \min_{z \in S} w(z).$$

$$(B) \quad w(u, S_u) = \frac{dw(\hat{A}(\tau))}{d\tau} > \theta \cdot w(\hat{A}(\tau)), \text{ where } \theta := \frac{\alpha-2}{k}.$$

Observe that because of the dummy pairs, we have $w(\hat{A}(\tau)) = 0$ for $\tau \in [0, k]$. Hence, statement (B) holds with equality for $\tau \in [0, k]$.

For contradiction's sake, we consider the infimum τ_0 over $\tau \geq k$ for which at least one of the above statements does not hold. Since all involved quantities are continuous in τ , one of the above statements does not hold for τ_0 .

CLAIM 4.1. *For all $B \subseteq \hat{A}(\tau_0)$ such that $\|B\| \leq k$, $w(\hat{S}(\tau_0)) \geq w(B)$.*

Proof. For $\tau < \tau_0$, statement (A) must hold. Hence, when S_u is increased while S_v is decreased, it must be the case that $w(u, S_u) \geq \frac{\alpha}{\alpha-1} \cdot w(v, S_v)$. This means the pair entering S has larger $w(\cdot)$ value than the pair leaving S . Therefore, it must be case that $w(\hat{S}(\tau_0))$ has the maximum $w(\cdot)$ value among all $B \subseteq \hat{A}(\tau_0)$ having measure $\|B\| = k$. ■

Hence, from the claim, we have $w(\hat{S}(\tau_0)) \geq w(\hat{A}(\tau_0)) - w(\hat{A}(\tau_0 - k))$.

Since statement (B) holds (maybe with equality) for $\tau < \tau_0$, by integrating from $\tau = \tau_0 - k$ to τ_0 , we have $w(\hat{A}(\tau_0)) \geq e^{\theta k} \cdot w(\hat{A}(\tau_0 - k))$. Therefore, $w(\hat{S}(\tau_0)) \geq (1 - e^{-\theta k}) \cdot w(\hat{A}(\tau_0))$.

Next, when $w(\hat{A})$ is about to increase, we must have some $u \in \Omega_l$ being considered such that $w(u, S_u) > \frac{1}{k} \cdot \{\alpha \cdot w(\hat{S}(\tau_0)) - w(\hat{A}(\tau_0))\} \geq \frac{1}{k} \cdot \{\alpha(1 - e^{-\theta k}) - 1\} \cdot w(\hat{A}(\tau_0)) = \theta \cdot w(\hat{A}(\tau_0))$,

where the last equality follows because $\theta = \frac{\alpha-2}{k}$ and $e^{\alpha-2} = \alpha$. Hence, statement (B) must hold.

Denote $\mathbf{m} := \min_{z \in \hat{S}(\tau_0)} w(z)$.

LEMMA 4.8. *For $0 \leq t \leq k \leq \tau_0$, $\mathbf{m} \leq \frac{w(\hat{S}(\tau_0 - t))}{k-t}$.*

Proof. We denote z_τ as the pair that is being added to $\hat{S}(\tau)$. Since at the same time, some pair may possibly be removed from $\hat{S}(\tau)$, we have $\frac{d}{dt} w(\hat{S}(\tau)) \leq w(z_\tau)$. Integrating this from $\tau = \tau_0 - t$ to τ_0 , we have

$$(4.4) \quad w(\hat{S}(\tau_0)) - w(\hat{S}(\tau_0 - t)) \leq w(\hat{A}(\tau_0)) - w(\hat{A}(\tau_0 - t)).$$

Denote $P := (\hat{A}(\tau_0) \setminus \hat{A}(\tau_0 - t)) \cap \hat{S}(\tau_0)$ and $Q := (\hat{A}(\tau_0) \setminus \hat{A}(\tau_0 - t)) \setminus P$. In other words, P is the set of pairs that arrive between $\tau_0 - t$ and τ_0 and still stay in

$\widehat{S}(\tau_0)$, and Q is the set of pairs arriving within the same time frame, but have been removed from \widehat{S} before τ_0 . Observe that $\|P \cup Q\| = t$.

Since pairs with minimum $w(\cdot)$ value are removed from \widehat{S} , we have for all $z \in Q$, $w(z) \leq m$. Hence, rearranging (4.4), we have

$$w(\widehat{S}(\tau_0 - t)) \geq w(\widehat{S}(\tau_0)) - w(P) - w(Q) \geq (k - t) \cdot m, \text{ as required.} \quad \blacksquare$$

Proving Statement (A). Define $\gamma := \frac{(\alpha-2)(\alpha-1)}{\alpha}$.

The easy case is when $w(\widehat{S}(\tau_0)) \leq \gamma \cdot w(\widehat{A}(\tau_0))$. Statement (B) implies that

$$w(u, S_u) > \theta \cdot w(\widehat{A}(\tau_0)) = \frac{\alpha}{\alpha-1} \cdot \frac{w(\widehat{S}(\tau_0))}{k} \geq \frac{\alpha}{\alpha-1} \cdot m, \text{ where the last inequality follows from Lemma 4.3 (with } t = 0).$$

From now on, we consider $w(\widehat{S}(\tau_0)) > \gamma \cdot w(\widehat{A}(\tau_0))$. We have

$$w(u, S_u) > \frac{1}{k} \cdot \{\alpha \cdot w(\widehat{S}(\tau_0)) - w(\widehat{A}(\tau_0))\} > \frac{1}{k} \cdot (\alpha\gamma - 1) \cdot w(\widehat{A}(\tau_0)).$$

Let $0 < t \leq \tau_0$ be the smallest t such that $w(\widehat{S}(\tau_0 - t)) \leq \gamma \cdot w(\widehat{A}(\tau_0 - t))$. We know such a t exists because $w(\widehat{S}(0)) = w(\widehat{A}(0)) = 0$.

Denoting z_τ as the pair that is being added to $\widehat{S}(\tau)$, we have for $\tau \in (\tau_0 - t, \tau_0]$,

$$(4.5) \quad \frac{d}{d\tau} w(\widehat{A}(\tau)) = w(z_\tau) > \frac{1}{k} \cdot (\alpha\gamma - 1) \cdot w(\widehat{A}(\tau)).$$

Define the function $\vartheta(x) := e^{(\alpha\gamma-1)x}(1-x)$ for $x \in [0, 1]$, and $\lambda := 1 - \frac{1}{\alpha\gamma-1}$, where $\lambda \in [0, 1]$ because $\alpha\gamma > 2$ (Fact 4.1 (a)). Observe that ϑ is increasing on $(0, \lambda)$ and decreasing on $(\lambda, 1)$. Hence, ϑ attains its maximum at λ and $\vartheta(\lambda) = \frac{1}{\alpha\gamma-1} \cdot e^{\alpha\gamma-2}$. We consider two cases.

Case 1. $t \leq \lambda k$. After integrating (4.5) on $\tau \in (\tau_0 - t, \tau_0]$, we have

$$w(\widehat{A}(\tau_0)) \geq \exp\{(\alpha\gamma - 1) \cdot \frac{t}{k}\} \cdot w(\widehat{A}(\tau_0 - t)).$$

Applying the definition of t , we have $w(\widehat{A}(\tau_0 - t)) \geq \frac{1}{\gamma} \cdot w(\widehat{S}(\tau_0 - t)) \geq \frac{k-t}{\gamma} \cdot m$, where the last inequality follows from Lemma 4.3.

Hence, in this case, we have

$$w(u, S_u) > \frac{\alpha\gamma-1}{\gamma} \cdot \vartheta(\frac{t}{k}) \cdot m \geq \frac{\alpha\gamma-1}{\gamma} \cdot \vartheta(0) \cdot m \geq \frac{\alpha}{\alpha-1} \cdot m,$$

where the last inequality follows from Fact 4.1 (b).

Case 2. $t > \lambda k$. After integrating (4.5) on $\tau \in (\tau_0 - \lambda k, \tau_0]$, we have

$$w(\widehat{A}(\tau_0)) \geq \exp\{(\alpha\gamma - 1) \cdot \lambda\} \cdot w(\widehat{A}(\tau_0 - \lambda k)).$$

Note that $w(\widehat{A}(\tau_0 - \lambda k)) \geq w(\widehat{S}(\tau_0 - \lambda k)) \geq (k - \lambda k) \cdot m$, where the last inequality follows from Lemma 4.3.

Hence, in this case, we have

$$w(u, S_u) > (\alpha\gamma - 1) \cdot \vartheta(\lambda) \cdot m = e^{\alpha\gamma-2} \cdot m \geq \frac{\alpha}{\alpha-1} \cdot m, \text{ where the last inequality comes from Fact 4.1 (c).}$$

FACT 4.1. (TECHNICAL INEQUALITIES) *The following*

inequalities can be verified easily, as the variables ($\alpha \approx 3.14619$ and $\gamma \approx 0.78188$) are absolute constants.

- (a) $\alpha\gamma > 2$.
- (b) $\frac{\alpha\gamma-1}{\gamma} \geq \frac{\alpha}{\alpha-1}$.
- (c) $e^{\alpha\gamma-2} \geq \frac{\alpha}{\alpha-1}$.

4.3 Rounding Procedure: Proof of Lemma 4.7

Recall that the goal is that given valid $S \in \mathcal{M}(\mathbb{R}_+^\Omega)$, we wish to show that $\widetilde{S} \subset \Omega$ produced by the rounding procedure in Definition 4.3 satisfies $\mathbf{E}[f(\widetilde{S})] \geq \widehat{f}(S)$.

As we shall see later, the procedure to obtain \widetilde{S} is related to sampling without replacement (in the limiting case) and the definition of \widehat{f} is related to independent sampling.

Sampling Distributions. Given a finite ground set \mathcal{U} , we define the following random subsets.

- (a) **Sampling without Replacement.** For an integer $k > 0$, denote $C_k(\mathcal{U})$ as the random subset obtained by sampling a k -subset from \mathcal{U} uniformly at random. In other words, it is sampling \mathcal{U} for k times without replacement.
- (a) **Independent Sampling.** Given $p \in [0, 1]$, denote $I_p(\mathcal{U})$ as the random subset obtained by including each element in \mathcal{U} independently with probability p .

LEMMA 4.9. (SAMPLING WITHOUT REPLACEMENT VS INDEPENDENT SAMPLING.) *Suppose $g : 2^{\mathcal{U}} \rightarrow \mathbb{R}^+$ is a submodular function. Moreover, $|\mathcal{U}| = n$ and $k \in \mathbb{Z}_+$ such that $p = \frac{k}{n} \in [0, 1]$. Then, we have*

$$\mathbf{E}[g(C_k(\mathcal{U}))] \geq \mathbf{E}[g(I_p(\mathcal{U}))].$$

Proof. For $0 \leq i \leq n$, define $g_i := \frac{1}{\binom{n}{i}} \sum_{T \in \binom{\mathcal{U}}{i}} g(T)$.

Observe that $g_i = \mathbf{E}[g(I_p(\mathcal{U})) \mid |I_p(\mathcal{U})| = i]$, since all subsets of size i are equally likely in independent sampling.

LEMMA 4.10. $g_i + g_{i-2} \leq 2g_{i-1}$, for all $2 \leq i \leq n$.

Proof. Define $\mathcal{N} := \{(P, Q) : P, Q \subseteq \mathcal{U}, |P| = |Q| = i-1, |P \cup Q| = i, |P \cap Q| = i-2\}$.

Observe that $N := |\mathcal{N}| = \binom{n}{i-2} \cdot (n-i+2) \cdot (n-i+1)$. By submodularity of g , we have the following

$$\sum_{(P,Q) \in \mathcal{N}} (g(P) + g(Q)) \geq \sum_{(P,Q) \in \mathcal{N}} (g(P \cup Q) + g(P \cap Q)).$$

Because of symmetry, subsets of \mathcal{U} with the same cardinality appear the same number of times. Hence, the inequality above becomes

$$2Ng_{i-1} \geq Ng_i + Ng_{i-2} \iff 2g_{i-1} \geq g_i + g_{i-2}.$$

\blacksquare

For $0 \leq i \leq n$, define $a_i := \Pr[|C_k(\mathcal{U})| = i]$ and $b_i := \Pr[|I_p(\mathcal{U})| = i]$. Observe that $a_k = 1$ and $b_i = \binom{n}{i} p^i (1-p)^{n-i}$.

Moreover, we have $\sum_{i=0}^n a_i = 1 = \sum_{i=0}^n b_i$ and $\sum_{i=0}^n i \cdot a_i = \mathbf{E}[|C_k(\mathcal{U})|] = k = np = \mathbf{E}[|I_p(\mathcal{U})|] = \sum_{i=0}^n i \cdot b_i$.

FACT 4.2. For any $\{c_i\}_{i=0}^n$ such that $\sum_{i=0}^n c_i = 0$ and $\sum_{i=0}^n i \cdot c_i = 0$, and any $\{g_i\}_{i=0}^n$, we have $\sum_{i=0}^n c_i g_i = \sum_{i=2}^n \sum_{j=i}^n (j-i+1) c_j \cdot (g_i + g_{i-2} - 2g_{i-1})$.

Proof. We write $g_{-1} = g_{-2} = 0$ and use the backward difference operator $\nabla g_i := g_i - g_{i-1}$.

Observing that $\sum_{j=0}^n c_j = \sum_{j=0}^n j c_j = 0$, we can add two initial terms to the RHS such that

$$\begin{aligned} \text{RHS} &= \sum_{i=0}^n \sum_{j=i}^n (j-i+1) \cdot c_j \cdot (\nabla g_i - \nabla g_{i-1}) \\ &= \sum_{j=0}^n \sum_{i=0}^j (j-i+1) \cdot c_j \cdot (\nabla g_i - \nabla g_{i-1}), \end{aligned}$$

where the last inequality follows from changing the order of summation. We next consider the coefficient of c_j as follows:

$$\begin{aligned} \sum_{i=0}^j (j-i+1)(\nabla g_i - \nabla g_{i-1}) &= \sum_{i=0}^j \nabla g_i - (j+1)\nabla g_{-1} \\ &= g_j, \end{aligned}$$

where the last equality follows from a telescoping sum and $g_{-1} = \nabla g_{-1} = 0$.

Hence, we have $\text{RHS} = \sum_{j=0}^n c_j g_j = \text{LHS}$, as required. \blacksquare

Hence, using the above fact, we have

$$\begin{aligned} \mathbf{E}[g(C_k(\mathcal{U}))] - \mathbf{E}[g(I_p(\mathcal{U}))] &= \sum_{i=0}^n (a_i - b_i) g_i \\ &= \sum_{i=2}^n \sum_{j=i}^n (j-i+1)(a_j - b_j) \cdot (g_i + g_{i-2} - 2g_{i-1}). \end{aligned}$$

Using Lemma 4.10, it suffices to show that $e_i := \sum_{j=i}^n (j-i+1)(a_j - b_j) \leq 0$ holds for all $i \geq 2$.

Observe that

$$e_i = \sum_{j=i}^n \Pr[|C_k(\mathcal{U})| \geq j] - \sum_{j=i}^n \Pr[|I_p(\mathcal{U})| \geq j].$$

For $i > k$, the first term in the above expression is 0. Hence, $e_i \leq 0$.

For $1 \leq i \leq k$, consider $e_i - e_{i+1} = \Pr[|C_k(\mathcal{U})| \geq i] - \Pr[|I_p(\mathcal{U})| \geq i] = 1 - \Pr[|I_p(\mathcal{U})| \geq i] \geq 0$.

Hence, it follows that for $2 \leq i \leq k$, $e_i \leq e_1 = \mathbf{E}[|C_k(\mathcal{U})|] - \mathbf{E}[|I_p(\mathcal{U})|] = 0$, as required. \blacksquare

LEMMA 4.11. (RESTATEMENT OF LEMMA 4.7) Given valid $S \in \mathcal{M}(\mathbb{R}_+^\Omega)$, the rounding procedure in Definition 4.3 generates \tilde{S} such that $\mathbf{E}[f(\tilde{S})] \geq \hat{f}(S)$.

Proof. Recall that we use $\tilde{\cdot}$ to represent the randomness used in the rounding procedure in Definition 4.3, and we use $R(\cdot)$ to represent the randomness used to define \hat{f} . Observe that for both $f(\tilde{S})$ and $\hat{f}(S)$, the randomness involved for different Ω_l 's are independent. We shall use a hybrid argument.

Fix $l \in [L]$. We condition on the randomness $\mathcal{R}_l := \bigcup_{l' \in \{1, \dots, l-1\}} R(S|_{\Omega_{l'}}) \cup \bigcup_{l' \in \{l+1, \dots, L\}} \widetilde{S|_{\Omega_{l'}}$. Define $f_l : 2^{\Omega_l} \rightarrow \mathbb{R}_+$ by $f_l(X) := f(\mathcal{R}_l \cup X)$. In order to apply the hybrid argument, it suffices to prove that for each $l \in [L]$,

$$(4.6) \quad \mathbf{E}[f_l(\widetilde{S|_{\Omega_l}})] \geq \mathbf{E}[f_l(R(S|_{\Omega_l}))].$$

Observe that the expectations on both sides of the inequality (4.6) are continuous in $S|_{\Omega_l}$. Hence, without loss of generality, we assume that for all $u \in \Omega_l$, S_u is rational. This means that for arbitrarily large n , we can form a partition \mathcal{U} of $S|_{\Omega_l}$ into n parts with equal measure such that each $x \in \mathcal{U}$ is associated with only one item in Ω_l . We write $k := k_l$ and each $x \in \mathcal{U}$ has measure $\frac{k}{n}$.

Define $g : 2^{\mathcal{U}} \rightarrow \mathbb{R}_+$ as $g(P) := f_l(\{u \in \Omega_l : \exists x \in P : x \text{ is associated with } u\})$. The submodularity of g follows from the submodularity of f_l (and f).

Define $\widehat{C}_k(\mathcal{U})$ to be a random sampling of \mathcal{U} for k times independently with replacement. Therefore, it follows that $\mathbf{E}[g(\widehat{C}_k(\mathcal{U}))] = \mathbf{E}[f_l(\widetilde{S|_{\Omega_l}})]$.

Let $p := \frac{k}{n}$. Fix some $u \in \Omega_l$ and let $r_u := \frac{S_u \cdot n}{k}$ be the number of elements in \mathcal{U} that are associated with u . It follows that the probability that at least one of these r elements appears in $I_p(\mathcal{U})$ is

$$\begin{aligned} 1 - (1-p)^{r_u} &\geq 1 - e^{-pr_u} = 1 - e^{-S_u} \\ &= \Pr[u \in R(S|_{\Omega_l})]. \end{aligned}$$

By the monotonicity of f , it follows that $\mathbf{E}[g(I_p(\mathcal{U}))] \geq \mathbf{E}[f_l(R(S|_{\Omega_l}))]$.

Let $\eta_n := \Pr[|\widehat{C}_k(\mathcal{U})| = k]$. As n tends to infinity, the probability of collision when sampling k items independently from a set of size n tends to 0. Hence, as n tends to infinity, η_n tends to 1.

Since g is non-negative, we have

$$\mathbf{E}[g(\widehat{C}_k(\mathcal{U}))] \geq \eta_n \cdot \mathbf{E}[g(C_k(\mathcal{U}))] \geq \eta_n \cdot \mathbf{E}[g(I_p(\mathcal{U}))],$$

where the last inequality follows from Lemma 4.9.

Finally, we have

$$\begin{aligned} \mathbf{E}\left[f_l(\widetilde{S}|\Omega_i)\right] &= \mathbf{E}\left[g(\widehat{C}_k(\mathcal{U}))\right] \geq \eta_n \cdot \mathbf{E}[g(l_p(\mathcal{U}))] \\ &\geq \eta_n \cdot \mathbf{E}[f_l(R(S|\Omega_i))]. \end{aligned}$$

Since this holds for arbitrarily large n , as n tends to infinity, we have the required result. \blacksquare

5 Hardness for Uniform Matroids

In this section, we give hardness results for deterministic monotone algorithms (satisfying Definitions 2.1) on uniform matroid constraints. Specifically, we show in the following theorem that the best ratio is $\frac{1}{\alpha_\infty}$, where $\alpha_\infty \in [3, 4]$ is the root of $\alpha = e^{\alpha-2}$.

THEOREM 5.1. *Suppose $\alpha \geq 1$ and $\alpha > e^{\alpha-2}$ (i.e., $\alpha < \alpha_\infty$). Then, there exists $k > 0$ such that with k -uniform matroid constraint, no deterministic monotone algorithm can have competitive ratio $\frac{1}{\alpha}$.*

Explanation. Before going into the details, we give an intuition on where the α_∞ comes from. The key insight is that in our hard instance, it suffices to compare $f(\text{OPT})$ with $f(\text{OPT} \cup A)$, if we consider strictly monotone algorithms. We consider an instance in which each arriving item is a subset of some “objects”, each of which has some non-negative weight. The objective function on a set of items is the weight of the union of the corresponding subsets of objects.

In each phase i , $2k$ distinct singleton items come, each containing an object with weight $w_i = (1 - \varepsilon)^{-i}$. Note that the weight grows exponentially. If the deterministic algorithm accepts $x_i \cdot k$ of them, then the adversary gives a “large” item which is the union of the $x_i \cdot k$ items the algorithm chooses in this phase. However, due to the monotonicity of the algorithm, this large item cannot be included into the solution, while it may appear in the OPT and only occupy one of the k quotas. Intuitively, a deterministic algorithm should exhibit convergent behavior after a large number of phases, in the sense that x_i converges to some x as i increases, because the algorithm faces essentially the same scenario in every phase.

Hence, after the n -th round,

$$\begin{aligned} f(S_n) &= kw_n \cdot x(1 + (1 - \varepsilon) + \cdots + (1 - \varepsilon)^{\frac{1}{x}-1}) \\ &= kw_n \cdot x \frac{1 - (1 - \varepsilon)^{\frac{1}{x}}}{\varepsilon} \approx kw_n \cdot \frac{x}{\varepsilon} (1 - e^{-\frac{\varepsilon}{x}}). \end{aligned}$$

On the other hand, $f(\text{OPT}_n)$ roughly equals $(k - n)w_n + kw_n \cdot x(1 + (1 - \varepsilon) + (1 - \varepsilon)^2 + \cdots)$, where the first term corresponds to $(k - n)$ singleton items in the last round and the second term corresponds to the “large”

items. Note that the “large” items actually captures A_n while each of them takes only one out of k quotas. This is what we mean by comparing to $f(\text{OPT} \cup A)$. When k is much larger than n , we have $f(\text{OPT}_n) \approx kw_n(1 + \frac{x}{\varepsilon})$.

Thus, the competitive ratio is bounded by

$$\frac{f(S_n)}{f(\text{OPT}_n)} = \frac{\frac{x}{\varepsilon}(1 - e^{-\frac{\varepsilon}{x}})}{1 + \frac{x}{\varepsilon}} \leq \frac{1}{\alpha_\infty},$$

where the inequality holds when $\frac{\varepsilon}{x} = \alpha_\infty - 2$.

One issue in the above sketch proof is that we consider k to be much larger than n , which we also assume to be large. To make the proof formal, we choose the parameters carefully. On a high level, assuming the existence of a $\frac{1}{\alpha}$ -competitive algorithm for all uniform matroids for some fixed α , the parameter k is chosen to be sufficiently large, and we only consider about δk phases for some small enough $\delta > 0$.

In the formal proof below, we first introduce some notations and give the construction of our instance. We assume the existence of a $\frac{1}{\alpha}$ -competitive strict monotone algorithm. This gives a family of constraints on the x_i variables since the algorithm has to maintain α ratio after every round. However, we don’t immediately have the property that the algorithm behaves the same in each round. Alternatively, we derive a lower bound for the x_i variables (Lemma 5.2) by induction and use it crucially to give a lower bound for α .

Parameters. Suppose $\varepsilon, \delta \in (0, 1)$ are parameters that can vary. For $i \geq 1$, define $w_i := (1 - \varepsilon)^{-i}$.

Ground Set Ω and Value Function f . In our construction, each element in the ground set Ω is a union of a finite number of bounded intervals in \mathbb{R}_+ . We define the function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $\varphi(x) := \sum_{i \geq 1} w_i \cdot \chi_{[i-1, i)}(x)$, i.e., if $i - 1 = \lfloor x \rfloor$, then $\varphi(x) = w_i$. Each element $A \in \Omega$ corresponds to a subset of $[0, +\infty)$. For a finite $S \subset \Omega$, the value function is $f(S) := 2 \int_{\cup_{A \in S} A} \varphi(x) dx$. That is, $f(S)$ is a weighted coverage function and, thus, is submodular.

Instance for k -Uniform Matroid. For each $k \geq 1$, we assume that there is an algorithm with competitive ratio $\frac{1}{\alpha}$. The instance depends on δ, ε and k . The next arriving element can be chosen adversarially depending on the algorithm’s previous action. Moreover, the adversary can stop at any moment, and hence the algorithm needs to maintain the ratio after every round. For $T := \lfloor \delta k \rfloor$, the elements arrive in T phases. For $1 \leq i \leq T$, the following happens in phase i .

- (a) There are $2k$ elements $\{[i - 1 + \frac{j-1}{2k}, i - 1 + \frac{j}{2k}) : j \in [2k]\}$ arriving one by one. Observe each is an interval in $[i - 1, i)$ with measure $\frac{1}{2k}$. Since in the construction these $2k$ elements in phase i are fixed, we can assume that if the algorithm selects an interval during phase i , then it will not discard

it before the next phase; otherwise, the algorithm needs not choose it in the first place. Moreover, if the algorithm needs to remove an interval from its feasible set, it will remove one from the earliest phase.

- (b) Suppose B_i is the collection of intervals selected by the algorithm in step (a). If B_i is non-empty, the next arriving element is the union of the intervals in B_i . Since the algorithm is strictly monotone, it will discard this element.

We write $x_0 := 0$. For $i \geq 1$, we define $x_i := \frac{|B_i|}{k}$ to be twice the measure of the union of intervals in B_i ; at the end of step (a) of phase i , denote S_i as the feasible set maintained by the algorithm, OPT_i as the current optimal solution and $A_i := \cup_{j \leq i} B_j$ as the intervals that have ever been picked by the algorithm so far.

Defining the sequence $\{\beta_m\}$. We next define a sequence $\{\beta_m\}_{m \geq 0}$ by $\beta_1 := \frac{\alpha-1}{1-\delta}$ and $\beta_{m+1} := \frac{\alpha-1-\alpha(1-\varepsilon)^{1+\frac{\beta_m}{\varepsilon}}}{1-\delta}$. Observe that the definition of the sequence depends only on α , ε and δ , and is independent of k and the algorithm.

LEMMA 5.1. ($\{\beta_m\}$ IS DECREASING) For $m \geq 1$, $\beta_{m+1} < \beta_m$.

Proof. We prove by induction on m . For $m = 1$, $\beta_2 = \frac{\alpha-1-\alpha(1-\varepsilon)^{1+\frac{\beta_1}{\varepsilon}}}{1-\delta} < \frac{\alpha-1}{1-\delta} = \beta_1$.

Suppose $\beta_m < \beta_{m-1}$ for some $m \geq 2$. We have

$$\begin{aligned} \beta_{m+1} < \beta_m &\Leftrightarrow \\ \alpha - 1 - \alpha(1-\varepsilon)^{1+\frac{\beta_m}{\varepsilon}} &< \alpha - 1 - \alpha(1-\varepsilon)^{1+\frac{\beta_{m-1}}{\varepsilon}} \\ \Leftrightarrow (1-\varepsilon)^{\frac{\beta_m-1}{\varepsilon}} &< (1-\varepsilon)^{\frac{\beta_{m-1}}{\varepsilon}} \\ \Leftrightarrow \beta_m < \beta_{m-1}, \end{aligned}$$

which is true by inductive hypothesis. This completes the inductive proof. \blacksquare

The following lemma is crucial to the hardness proof. Even though the definition of the β_m 's is independent of the algorithm, we use the assumption on the algorithm's competitive ratio to place constraints on the x_i 's and infer that each β_m is positive.

LEMMA 5.2. ($\{\beta_m\}$ IS POSITIVE) For $m \geq 1$, $\beta_m > 0$.

Constraints on x_i 's. Suppose $\mathbf{x} := \{x_i\}_{i=1}^T$, and for notational convenience, we write $x_0 = 1$ and $w_0 = 0$. For $1 \leq n \leq T$, define $i_n = i_n(\mathbf{x})$ to be the smallest index such that $\sum_{i=i_n}^n x_i < 1$; if $x_n = 1$, set $i_n = n+1$, and we interpret the summation $\sum_{i=i_n+1}^n$ as an empty sum equal to zero. Then, the value of the feasible set at the end of phase n is

$$f(S_n(\mathbf{x})) = \sum_{i=i_n}^n x_i w_i + (1 - \sum_{i=i_n}^n x_i) w_{i_n-1}.$$

On the other hand, another feasible solution is to take the sets in all the step (b)'s from phase 1 to phase n , together with $k-n$ sets in step (a) of phase n . Hence, $f(\text{OPT}_n) \geq \sum_{i=1}^n x_i w_i + (1 - \frac{n}{k}) \cdot w_n \geq \sum_{i=1}^n x_i w_i + (1 - \delta) w_n$. Since the algorithm has competitive ratio $\frac{1}{\alpha}$, we have $\alpha \cdot f(S_n) \geq f(\text{OPT}_n)$. Hence, we have shown that given any $k > 0$, for $T = \lfloor \delta k \rfloor$, there exists a sequence of numbers $\mathbf{x} = \{x_i\}_{i=1}^T$ in $[0, 1]$ satisfying the following:

$$\begin{aligned} \forall n \in [T], \alpha \cdot f(S_n(\mathbf{x})) \\ &= \alpha \left(\sum_{i=i_n}^n x_i w_i + (1 - \sum_{i=i_n}^n x_i) w_{i_n-1} \right) \\ (5.7) \quad &\geq \sum_{i=1}^n x_i w_i + (1 - \delta) w_n. \end{aligned}$$

The following lemma allows us to assume that all equalities in (5.7) hold.

LEMMA 5.3. Suppose there exists a solution $\{x_i\}_{i=1}^T$ for (5.7). Then, there exists a solution $\{x'_i\}_{i=1}^T$ such that all equalities hold.

Proof. Suppose n is the smallest index such that the inequality in (5.7) is strict. We will show that the n -th inequality can be made into equality by decreasing x_n and perhaps increasing x_{n+1} . Since inequalities with indices smaller than n do not involve x_n or x_{n+1} , those equalities will be maintained. On the other hand, we will show that inequalities with indices larger than n will not be violated. Hence, we can go through the inequalities from smaller to larger indices to transform all strict inequalities into equalities.

Fixing the values of x_1, x_2, \dots, x_{n-1} , we consider the difference of both sides of the n -th inequality as a function of x_n given by:

$$h(x) := \alpha \cdot f(S_n(\mathbf{x}_{[n-1]}, x)) - \sum_{i=1}^{n-1} x_i - x - (1 - \delta) w_n,$$

which is continuous.

From our assumption, $h(x_n) > 0$; on the other hand, $h(0) = \alpha \cdot f(S_{n-1}(\mathbf{x}_{[n-1]})) - \sum_{i=1}^{n-1} x_i - (1 - \delta) w_n < \alpha \cdot f(S_{n-1}(\mathbf{x}_{[n-1]})) - \sum_{i=1}^{n-1} x_i - (1 - \delta) w_{n-1} = 0$, where the last equality holds from the choice of n . Therefore, $h(x) = 0$ for some $x \in (0, x_n)$; we let \hat{x} to be the largest number in $(0, x_n)$ such that $h(\hat{x}) = 0$.

Stage 1: $x_{n+1} < 1$. We decrease x_n and increase x_{n+1} continuously such that $w_{n+1} x_{n+1} + w_n x_n$ remains constant. This stage ends when x_{n+1} reaches 1 or x_n reaches \hat{x} , whichever happens first. (If the latter happens first, then there is no need for Stage 2.)

As remarked above, all inequalities with indices smaller than n are not affected and so they remain equalities. Consider the m -th inequality, where $m \geq n+1$. Observe that the right hand side is $\sum_{i=1}^m x_i w_i + (1-\delta)w_n$, which does not change. Hence, it suffices to show that the left hand side does not decrease.

Observe that $f(S_m(\mathbf{x})) = \sum_{i=i_m}^m x_i w_i + (1 - \sum_{i=i_m}^m x_i)w_{i_m-1}$.

We consider the following cases. We remark that i_m could change during Stage 1.

- (a) Case $i_m \geq n+2$. In this case, $f(S_m(\mathbf{x}))$ is independent of x_n and x_{n+1} and so does not change.
- (b) Case $i_m = n+1$. In this case, $f(S_m(\mathbf{x}))$ depends only on x_{n+1} . As x_{n+1} increases at rate 1, $f(S_m(\mathbf{x}))$ increases at rate $w_{n+1} - w_n$. However, observe that as x_{n+1} increases and x_n decreases, i_m could change from $n+1$ to n .
- (c) Case $i_m \leq n$. In this case, the first term $\sum_{i=i_m}^m x_i w_i$ does not change. However, as x_n decreases and x_{n+1} increases to keep $w_{n+1}x_{n+1} + w_n x_n$ constant, it follows that $x_n + x_{n+1}$ decreases. Hence, the second term $(1 - \sum_{i=i_m}^m x_i)w_{i_m-1}$ increases. Observe that this could cause i_m to further decrease, but $f(S_m(\mathbf{x}))$ never decreases.

Stage 2: $x_{n+1} = 1$. Suppose x_{n+1} reaches 1 first before x_n reaches \hat{x} . When this happens, we keep x_{n+1} at 1 and only decreases x_n (continuously) to \hat{x} . Consider the m -th inequality where $m \geq n+1$. Observe that since $x_{n+1} = 1$, $i_m \geq n+2$, and hence the left hand side does not change. On the other hand, as x_n decreases, the right hand side decreases. Therefore, the m -th inequality is not violated.

This completes the proof of Lemma 5.3. \blacksquare

Proof of Lemma 5.2: We prove the following stronger statement. Define $N_m = 1 + \sum_{i=1}^{m-1} \lceil \frac{\beta_i}{\varepsilon} \rceil$. Suppose k is sufficiently large such that $T = \lceil \delta k \rceil > N_m$, and $\{x_i\}_{i=1}^T$ is a sequence in $[0, 1]$ satisfying all equalities in (5.7). Then, for all $N_m < n \leq T$, $\beta_m x_n \geq \varepsilon$ (which implies that $\beta_m > 0$). We prove this by induction on m .

For $m = 1$ and $n > N_1 = 1$, from $f(S_n(\mathbf{x})) \leq f(S_{n-1}(\mathbf{x})) + x_n w_n$, we use equalities in (5.7) to derive the following.

$$\begin{aligned} \sum_{i=1}^n x_i w_i + (1-\delta)w_n &= \alpha \cdot f(S_n(\mathbf{x})) \\ &\leq \alpha \cdot f(S_{n-1}(\mathbf{x})) + \alpha x_n w_n \\ &= \sum_{i=1}^{n-1} x_i w_i + (1-\delta)w_{n-1} + \alpha x_n w_n, \end{aligned}$$

which is equivalent to $(\alpha - 1)x_n w_n \geq (1 - \delta)(w_n -$

$w_{n-1}) = (1 - \delta) \cdot \varepsilon w_n$. Rearranging gives $\beta_1 x_n \geq \varepsilon$.

Now suppose that for some $m \geq 1$, for all $N_m < i \leq T$, $\beta_m x_i \geq \varepsilon$.

Consider $T \geq n > N_m + \lceil \frac{\beta_m}{\varepsilon} \rceil$. Then, for $1 \leq j \leq \lceil \frac{\beta_m}{\varepsilon} \rceil$, $x_{n-j} \geq \frac{\varepsilon}{\beta_m}$. Hence, it follows that $\sum_{j=1}^{\lceil \frac{\beta_m}{\varepsilon} \rceil} x_{n-j} \geq 1$. Therefore, $i_{n-1} \geq n - \lceil \frac{\beta_m}{\varepsilon} \rceil + 1$.

Observe that in transforming the solution from S_{n-1} to S_n , elements associated with w_j is replaced by elements associated with w_n , where $j \geq n - \lceil \frac{\beta_m}{\varepsilon} \rceil$. Hence, we have $f(S_n(\mathbf{x})) \leq f(S_{n-1}(\mathbf{x})) + x_n \cdot (w_n - w_{n-\lceil \frac{\beta_m}{\varepsilon} \rceil})$. Again, using equalities (5.7), we have:

$$\begin{aligned} \sum_{i=1}^n x_i w_i + (1-\delta)w_n &= \alpha \cdot f(S_n(\mathbf{x})) \\ &\leq \alpha \cdot f(S_{n-1}(\mathbf{x})) + \alpha x_n \cdot (w_n - w_{n-\lceil \frac{\beta_m}{\varepsilon} \rceil}) \\ &= \sum_{i=1}^{n-1} x_i w_i + (1-\delta)w_{n-1} \\ &\quad + \alpha x_n \cdot (w_n - w_{n-\lceil \frac{\beta_m}{\varepsilon} \rceil}). \end{aligned}$$

Rearranging gives $\beta_{m+1} x_n \geq \frac{\alpha - 1 - \alpha(1-\varepsilon)^{\lceil \frac{\beta_m}{\varepsilon} \rceil}}{1-\delta} \cdot x_n \geq \varepsilon$, completing the inductive proof. \blacksquare

Completing the Proof of Theorem 5.1. Recall that for some $\alpha \geq 1$, we assume that for any $k \geq 0$, there is a deterministic monotone algorithm for the k -uniform matroid with competitive ratio $\frac{1}{\alpha}$. Then, for any $\delta, \varepsilon \in (0, 1)$, we define a sequence $\{\beta_m\}_{m \geq 1}$ (depending on only α, δ and ε). In Lemma 5.1, we show that the sequence is decreasing. In Lemma 5.2, using the assumption on the competitive ratio of the algorithm, we show that each β_m is positive. Hence, by the monotone convergence theorem, the sequence converges to some limit β , which satisfies the following equation: $\beta = \frac{\alpha - 1 - \alpha(1-\varepsilon)^{1+\frac{\beta}{\varepsilon}}}{1-\delta}$. After rearranging, we have $\alpha - 1 = g(\beta)$, where

$$g(t) := t(1-\delta) + \alpha(1-\varepsilon)^{1+\frac{t}{\varepsilon}}.$$

Then, $g'(t) = (1-\delta) + \alpha \cdot \frac{\ln(1-\varepsilon)}{\varepsilon} \cdot (1-\varepsilon)^{1+\frac{t}{\varepsilon}}$.

Writing $c(\varepsilon) := -\frac{\varepsilon}{\ln(1-\varepsilon)}$, g attains its minimum when $g'(t) = 0$, i.e., $(1-\varepsilon)^{1+\frac{t}{\varepsilon}} = \frac{c(\varepsilon) \cdot (1-\delta)}{\alpha}$.

Hence, we have $\alpha - 1 = g(\beta) \geq (1-\delta) \cdot (c(\varepsilon) \cdot \ln \frac{\alpha}{c(\varepsilon) \cdot (1-\delta)} + c(\varepsilon) - \varepsilon)$, where the inequality holds for all $\varepsilon, \delta \in (0, 1)$.

Since the relevant quantities are all continuous in ε and δ , as ε and δ tend to zero, $c(\varepsilon)$ tends to 1, and the above inequality becomes $\alpha - 1 \geq \ln \alpha + 1$, which is equivalent to $e^{\alpha-2} \geq \alpha$, as required. \blacksquare

6 Hardness for Partition Matroids

In this section, we give hardness results for deterministic algorithms on partition matroids. Specifically, the ground set $\Omega := \cup_{i \geq 0} \Omega_i$ is a union of disjoint sets Ω_i 's such that a (finite) set S is independent iff for all i , $|S \cap \Omega_i| \leq 1$.

We consider a universe \mathcal{U} of items, each of which has a weight given by $\nu : \mathcal{U} \rightarrow \mathbb{R}_+$. A subset $X \subseteq \mathcal{U}$ has weight $\nu(X) := \sum_{x \in X} \nu(x)$. Then, for $i \geq 0$, $\Omega_i := \{(u, i) : u \in \mathcal{U}\}$. For $S \subset \Omega$, we define $f(S) := \nu(\{x : (x, i) \in S\})$.

6.1 Hardness for Monotone Algorithms We show that for general matroids, the competitive ratio $\frac{1}{4}$ is optimal for monotone algorithms satisfying Definition 2.1 (recalling that a monotone algorithm that achieves this ratio is given in the full version[14]). In particular, we show the following hardness result.

THEOREM 6.1. *For any $\alpha < 4$, no monotone deterministic algorithm can have competitive ratio **strictly larger than** $\frac{1}{\alpha}$.*

Adversarial Model. Given any $1 \leq \alpha < 4$, we construct a finite sequence of elements. For any algorithm, an adversary can adaptively decide when to stop the arrival of items, at which moment the competitive ratio will be at most $\frac{1}{\alpha}$.

Instance Construction. Given $\alpha < 4$, we shall pick some large enough n (to be decided later), and consider $\Omega := \cup_{i=0}^n \Omega_i$.

The sequence of elements come in n phases. For $1 \leq i \leq n$, in phase i , two elements arrive in the order: $(x_i, 0)$, (x_i, i) , for some $x_i \in \mathcal{U}$. We shall define the weights of the $a_i := \nu(x_i)$ carefully. If the algorithm does not take the element $(x_i, 0)$, then the adversary stops the sequence, and we shall see the competitive ratio will be at most $\frac{1}{\alpha}$. However, if the algorithm takes $(x_i, 0)$, then it cannot take the next element (x_i, i) due to strict monotonicity.

Defining $a_i = \nu(x_i)$ and the invariant. We choose $a_1 = 1$. Recall that $1 \leq \alpha < 4$. We show that if the algorithm has competitive ratio strictly greater than $\frac{1}{\alpha}$, then the following invariant holds: after phase i , the algorithm will have selected $(x_i, 0)$. As described above, this is true after phase 1.

After phase i , the value achieved by the algorithm is a_i under f , while the optimal solution is $\{(x_j, j) : j \in [i]\}$ having value $\sum_{j=1}^i a_j$. We next define the weight $a_{i+1} = \nu(x_{i+1})$ such that the following holds:

$$\sum_{j=1}^{i+1} a_j = \alpha a_i.$$

For instance, if α is just a little less than 4, then a_2 is close to 3.

In phase $(i+1)$, the element $(x_{i+1}, 0)$ arrives first. If the algorithm does not take it, then the adversary stops the sequence. In this case, the algorithm has only selected $(x_i, 0)$, whose value is $a_i = \frac{1}{\alpha} \cdot \sum_{j=1}^{i+1} a_j$, which is at most $\frac{1}{\alpha}$ fraction of the optimal value. Hence, the algorithm must replace $(x_i, 0)$ with $(x_{i+1}, 0)$. As described above, a monotone algorithm cannot take the next element $(x_{i+1}, i+1)$. Hence, we show that the invariant holds after phase $(i+1)$.

Choosing n . We next show that there exists some n such that after phase n , the competitive ratio is strictly less than $\frac{1}{\alpha}$. Observe that the weights a_i 's are determined totally by the recursion: $a_1 = 1$ and $a_{i+1} = \alpha a_i - \sum_{j=1}^i a_j$.

By considering the difference of the recursive definitions of a_{i+2} and a_{i+1} , we can obtain the following second order recursion: $a_{i+2} - \alpha a_{i+1} + \alpha a_i = 0$.

Since $\Delta = \alpha^2 - 4\alpha < 0$, the characteristic equation has complex roots. By Lemma 6.1, this sequence will eventually return a negative number. We can pick n to be the smallest integer such that $a_{n+1} < 0$. Hence, after phase n , the algorithm has value $a_n = \frac{1}{\alpha} \sum_{j=1}^{n+1} a_j < \frac{1}{\alpha} \sum_{j=1}^n a_j$, which is $\frac{1}{\alpha}$ fraction of the optimal value. Hence, to complete the hardness proof, it suffices to show the following lemma.

LEMMA 6.1. *Suppose $P, Q > 0$ such that $\Delta := P^2 - 4Q < 0$. The sequence $\{a_n\}$ is defined by the recursion $a_{n+2} - Pa_{n+1} + Qa_n = 0$ where both a_1 and a_2 are real and at least one is non-zero. Then, there exists $n > 0$ such that $a_n < 0$.*

Proof. Since $\Delta < 0$, the characteristic equation $\rho^2 - P\rho + Q = 0$ has complex roots ρ and $\bar{\rho}$. Since $\text{Re}(\rho) = \frac{P}{2} > 0$, we can write $\rho = re^{i\phi}$, where $r > 0$ and $0 < \phi < \frac{\pi}{2}$.

A standard technique to solve recurrence relation gives that a_n is a linear combination of $\rho^n = r^n e^{in\phi}$ and $\bar{\rho}^n = r^n e^{-in\phi}$ [38]. Since a_n is real, it follows that there exist real numbers A and B such that $a_n = r^n (A \cos n\phi + B \sin n\phi)$. Since at least one of a_0 and a_1 is non-zero, at least one of A and B is non-zero.

Finally, since $0 < \phi < \frac{\pi}{2}$, as n increases, $n\phi$ will eventually reach all of the following 4 intervals: $(0, \frac{\pi}{2})$, $(\frac{\pi}{2}, \pi)$, $(\pi, \frac{3\pi}{2})$, $(\frac{3\pi}{2}, 2\pi)$. Hence, there exists $n > 0$ such that $\cos n\phi$ has opposite sign as A (if $A \neq 0$) and $\sin n\phi$ has opposite sign as B (if $B \neq 0$), which implies that $a_n < 0$, as required. ■

6.2 Hardness for General Deterministic Algorithms Similar to Section 6.1, we show that for algorithms that are not necessarily monotone, the competi-

tive ratio cannot be better than $\frac{3-\sqrt{5}}{2} \approx 0.382$. Specifically, we also use a partition matroid and show the following.

THEOREM 6.2. *For any $\alpha < \frac{3+\sqrt{5}}{2}$, no deterministic algorithm can have competitive ratio **strictly larger than** $\frac{1}{\alpha}$.*

Adversarial Model. Fix $1 \leq \alpha < \frac{3+\sqrt{5}}{2}$. Unlike the case in Section 6.1, the sequence of arriving items will adapt according to the action of the algorithm. The elements arrive in phases. For $i \geq 1$, in phase i , the following steps happen.

- (a) First, $x_i^{(0)}$ and $x_i^{(1)}$ are distinct items in \mathcal{U} with the same value $a_i = \nu(x_i^{(0)}) = \nu(x_i^{(1)})$ to be decided later. The elements $(x_i^{(0)}, 0)$ and $(x_i^{(1)}, 0)$ in Ω_0 arrive (one after another). We shall show that the algorithm must select at least one of them, say $(x_i^{(\chi_i)}, 0)$, for $\chi_i \in \{0, 1\}$. Otherwise, the adversary terminates the sequence.
- (b) Next, there is some item $y_i \in \mathcal{U}$ with value $b_i = \nu(y_i)$ to be decided later. Then, the elements $(y_i, 2i - 1)$ and $(x_i^{(\chi_i)}, 2i - 1)$ in Ω_{2i-1} arrive (one after another), where $(x_i^{(\chi_i)}, 0)$ is the element selected by the algorithm in step (a).
- (c) If the algorithm selects an element $(z, 2i - 1)$ in step (b), choose $\hat{z} := z$, then the element $(z, 2i) \in \Omega_{2i}$ arrives; otherwise, choose $\hat{z} := x_i^{(\chi_i)}$. If z is $x_i^{(\chi_i)}$, then the adversary terminates the sequence.

Invariant. We show that if the algorithm has competitive ratio **strictly larger than** $\frac{1}{\alpha}$, then after each step in phase i , the following holds.

- (a) The feasible set maintained by the algorithm contains $\{(y_j, 2j - 1) : j \in [i - 1]\}$ and some $(x_i^{(\chi_i)}, 0)$. The f value achieved by the algorithm is $\sum_{j=0}^{i-1} b_j + a_i$, while the optimal value is $\sum_{j=0}^{i-1} (a_j + b_j) + a_i$ attained by the solution $\{(x_j^{(\chi_j)}, 2j - 1), (y_j, 2j) : j \in [i - 1]\} + (x_i^{(1-\chi_i)}, 0)$.
- (b) The algorithm selects $(y_i, 2i - 1)$.
- (c) The feasible set maintained by the algorithm contains $\{(y_j, 2j - 1) : j \in [i]\}$, $(x_i^{(\chi_i)}, 0)$ and $(y_i, 2i - 1)$. The optimal solution is $\{(x_j^{(\chi_j)}, 2j - 1), (y_j, 2j) : j \in [i]\} + (x_i^{(1-\chi_i)}, 0) + (y_i, 2i)$.

Defining a_i and b_i to maintain the invariant. We define $a_1 := 1$. Observe that in step (a) of phase 1, the algorithm must pick at least 1 element. Otherwise, the algorithm has value 0, while the optimal value is 1.

Inductive argument. We assume that for some $i \geq 1$, the invariant holds up to the moment after step (a) of phase i , when $\{a_j\}_{j=1}^i$ and $\{b_j\}_{j=1}^{i-1}$ are already defined.

We shall show that the invariant continues to hold after step (a) of phase $i + 1$, and define b_i and a_{i+1} . We define b_i such that the following holds:

$$(6.8) \quad \alpha(a_i + \sum_{j=1}^{i-1} b_j) = \sum_{j=1}^i a_j + a_i + \sum_{j=1}^i b_j.$$

If $b_i \leq 0$, the adversary can terminate immediately, because $\alpha(a_i + \sum_{j=1}^{i-1} b_j) \leq \sum_{j=1}^i a_j + a_i + \sum_{j=1}^{i-1} b_j$, which is the optimal value achieved at the end of step (a) of phase i . However, at this moment, the algorithm has value $a_i + \sum_{j=1}^{i-1} b_j$, and so the competitive ratio is at most $\frac{1}{\alpha}$. Hence, we can assume $b_i > 0$.

Observe that the algorithm must take $(y_i, 2i - 1)$ in step (b). Otherwise, the sequence terminates after step (c), and the right hand side of (6.8) is the optimum value. The algorithm attains value $a_i + \sum_{j=1}^{i-1} b_j$, and hence has competitive ratio $\frac{1}{\alpha}$. However, recall that we assume the algorithm has ratio strictly larger than $\frac{1}{\alpha}$.

Hence, the algorithm selects $(y_i, 2i - 1)$ and it does not help to select $(y_i, 2i)$ in step (c). However, the optimal solution could include $(x_i^{(1-\chi_i)}, 0)$ in step (a), $(x_i^{(\chi_i)}, 2i - 1)$ in step (b) and $(y_i, 2i)$ in step (c).

We next consider the beginning of step (a) in phase $i + 1$. We define a_{i+1} such that the following holds:

$$(6.9) \quad \alpha(a_i + \sum_{j=1}^i b_j) = \sum_{j=1}^{i+1} a_j + \sum_{j=1}^i b_j.$$

Observe that subtracting (6.8) from (6.9) gives $a_{i+1} - a_i = \alpha b_i > 0$. Hence, the optimal solution will replace the old element in Ω_0 arrived in phase i with one of the new elements arrived in step (a) of phase $i + 1$.

Moreover, the algorithm must select some element $(x_{i+1}^{(\chi_{i+1})}, 0)$. Otherwise, the sequence terminates, and the right hand side of (6.9) is the optimal value, while the algorithm achieves value $a_i + \sum_{j=1}^i b_j$, which is exactly $\frac{1}{\alpha}$ fraction of the optimal value. Recall that the algorithm should achieve ratio strictly larger than $\frac{1}{\alpha}$.

This completes the inductive argument and the recursive definitions of a_i and b_i .

There exists negative b_n . As in Section 6.1, we show that there exists $n > 0$ such that $b_n < 0$. Suppose n is the smallest integer such that this happens. Then, in the above inductive argument, it follows that after step (a) of phase n , the competitive ratio is strictly less than $\frac{1}{\alpha}$.

To apply Lemma 6.1, we shall form a second order recurrence relation for b_n . If we consider (6.8) minus (6.9) ($i \leftarrow i - 1$), we have: $\alpha(a_i - a_{i-1}) = a_i + b_i$. From the inductive argument, we have $a_i - a_{i-1} = \alpha b_{i-1}$. Hence, the above equation becomes $\alpha^2 b_{i-1} = a_i + b_i$.

Replacing i with $i + 1$ gives $\alpha^2 b_i = a_{i+1} + b_{i+1}$. Taking the difference between the last two equations gives:

$$\begin{aligned}\alpha^2(b_i - b_{i-1}) &= (a_{i+1} - a_i) + b_{i+1} - b_i \\ &= \alpha b_i + b_{i+1} - b_i.\end{aligned}$$

Rearranging gives the required second order recurrence relation on b_i :

$$b_{i+1} - (\alpha^2 - \alpha + 1)b_i + \alpha^2 b_{i-1} = 0.$$

To apply Lemma 6.1, observe that $P := \alpha^2 - \alpha + 1 > 0$, $Q := \alpha^2$, and $\Delta := (\alpha^2 - \alpha + 1)^2 - 4\alpha^2 = (\alpha^2 + \alpha + 1)(\alpha^2 - 3\alpha + 1)$, which is negative for $1 \leq \alpha < \frac{3+\sqrt{5}}{2}$, as required.

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